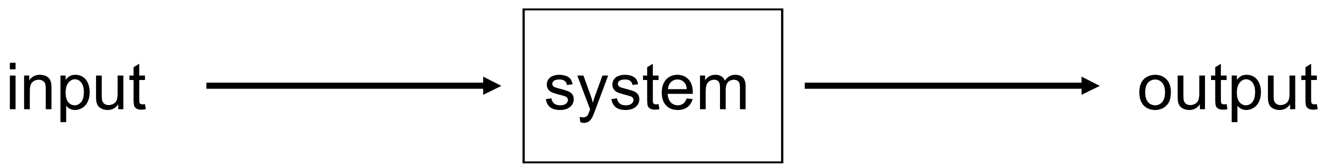
system theory of imaging systems

system theory:

provides mathematical tools to allow transformation of a physically encoded information into another representation without loss of Information (e.g. from position-space to Fourier space)

transmission system:



examples:

1D encoded information:

input = language; system = telephone; output = acoustic signal (phone)
information: time-variant membrane pressure

2D encoded information:

input = image; system = xerox machine; output = copy of image
information: location-dependent grey level distribution

system theory of imaging systems

transmission system = imaging system

input → system → output

f(x,y)	imaging system	g(x,y)
x-ray dose D(x,y)	x-ray system with amplification film	film
attenuation coefficient μ(x,y)	CT-system	digitized image
proton density ρ(x,y)	MRI-system	image on monitor

system theory of imaging systems

definitions **system properties**

an imaging system with:

$f_i(x,y)$ → System → $g_i(x,y)$

is called **linear**, iff:

$\Sigma c_i f_i(x,y)$ → System → $\Sigma c_i g_i(x,y)$

an imaging system with:

$f(x,y)$ → System → $g(x,y)$

is called **translation-invariant**, iff:

$f(x-x_0, y-y_o)$ → System → $g(x-x_0, y-y_o)$

system theory of imaging systems

definitions

mathematical methods for system characterization:

- **Dirac function**
- **Fourier transform and convolution theorem**
- **time domain:**
  - **impulse response / transfer function**
- **spatial domain:**
  - **point spread function / modulation transfer function**
- **auto-/cross-correlation function**

additional aspects for real systems:

- **noise**
- **sampling; aliasing**
- **filtering**

definitions **1D Dirac function**

with rectangular function $\text{rect}(t) = \begin{cases} 1 \text{ für } |t| \leq 1/2 \\ 0 \text{ für } |t| > 1/2 \end{cases}$

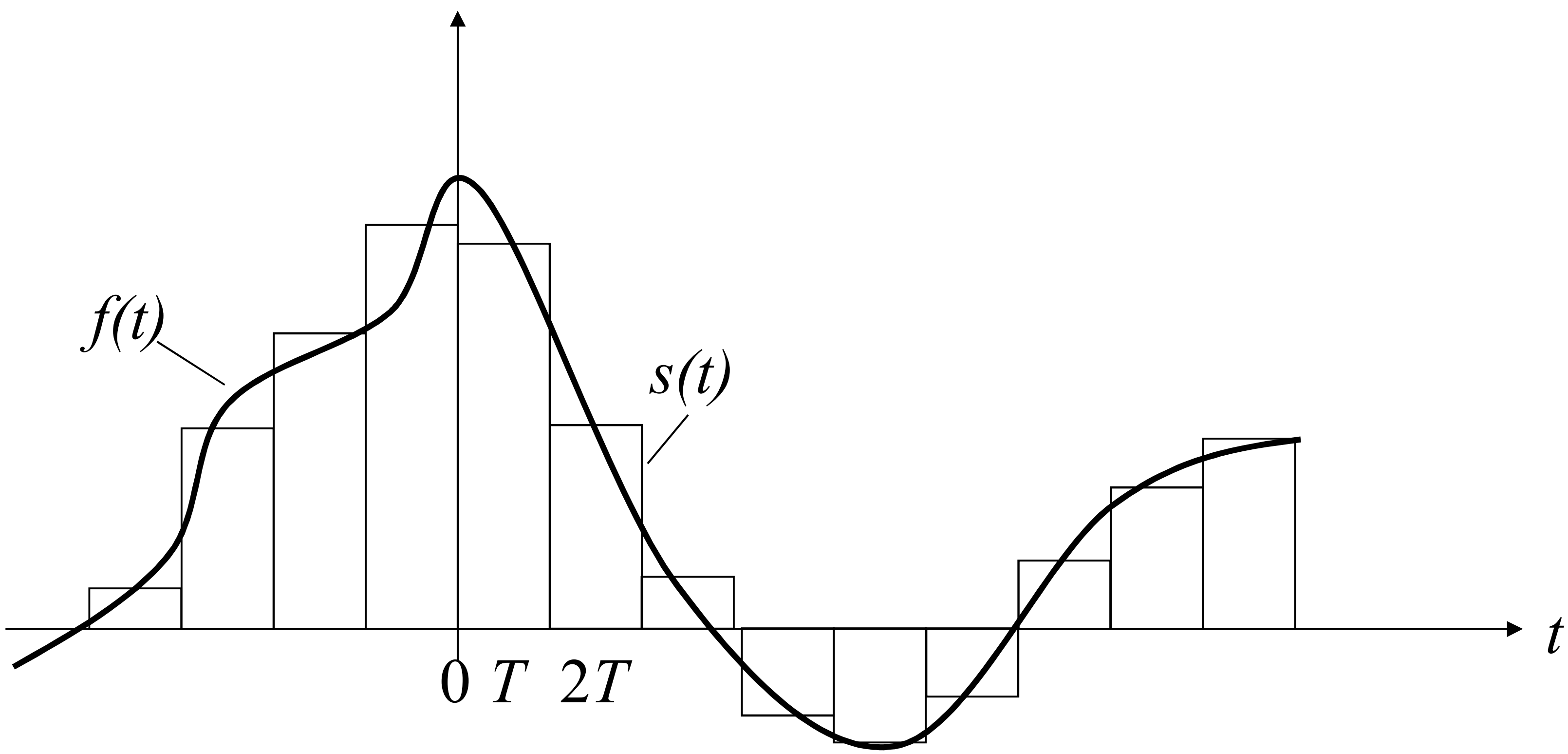
follows the definition of δ**-function** (Dirac function):

$$\delta(t) := \lim_{T \to 0} \frac{1}{T} \cdot \text{rect}\left(\frac{t}{T}\right) = \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

δ-function: infinitely short pulse with infinitely large amplitude

definitions **1D Dirac function**

approximation of some function $f(t)$ with sequence of rect-functions



definitions **1D Dirac function**

- approximation of some function $f(t)$ with sequence of rect-functions

- the smaller T, the more accurate the approximation

- for $T \rightarrow 0$:

$$n \cdot T \rightarrow \tau, T \rightarrow d\tau, \lim_{T \rightarrow 0} s(t) = \delta(t)$$

$$\lim_{T \rightarrow 0} s(t) = f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$

definitions **1D Dirac function**

approximation of some function $f(t)$ with sequence of rect-functions

the integral $f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$ is called **convolution**

of function f with Dirac function und can be written as:

$$f(t) * \delta(t) = f(t)$$

definitions **2D Dirac function**

analogue definitions for 2D case

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(p,q)\delta(x-p,y-q)\,dp\,dq$$

$$=f(x,y)*\delta(x,y)$$

$\delta(x,y)$ is a two-dimensional impulse

definitions **Dirac function**

properties of the δ-function:

filtering: $$f(x_0) = \int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx$$

linearity: $$c_1 \cdot \delta(x) + c_2 \cdot \delta(x) = (c_1 + c_2) \cdot \delta(x)$$

symmetry: $$\delta(-x) = \delta(x)$$

elongation: $$\delta(bx) = \frac{1}{|b|} \cdot \delta(x)$$

definitions **convolution**

properties of convolution algebra:

convolution $$g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(y) h(x-y) dx$$

identity $$f(x) = f(x) * \delta(x) = \delta(x) * f(x)$$

commutative- $$f(x) * h(x) = h(x) * f(x)$$

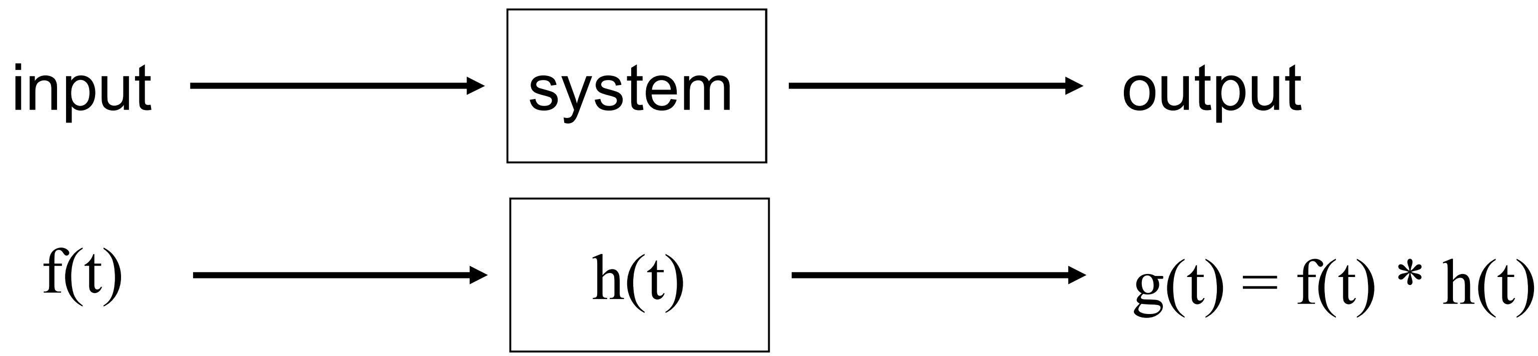
associative- $$\left[f(x) * g(x)\right] * h(x) = f(x) * \left[g(x) * h(x)\right]$$

distributive-law/linearity $$f(x) * \left[c_1 h_1(x) + c_2 h_2(x)\right] = c_1 f(x) * h_1(x) + c_2 f(x) * h_2(x)$$

differentiation $$\left(f(x) * h(x)\right)' = f'(x) * h(x) = f(x) * h'(x)$$

system theory of imaging systems

definitions **Fourier transform**

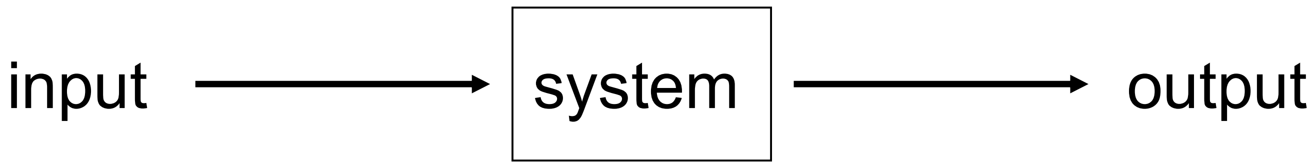


- in signal processing $h(t)$ is called **impulse response**

- for $h(t) = \delta(t)$, the system is called **ideally distortion-free** since $f(t) = \delta(t) * f(t)$ holds

system theory of imaging systems

definitions **Fourier transform**



- consider input functions whose amplitudes are influenced by the system, but there are no other changes of form

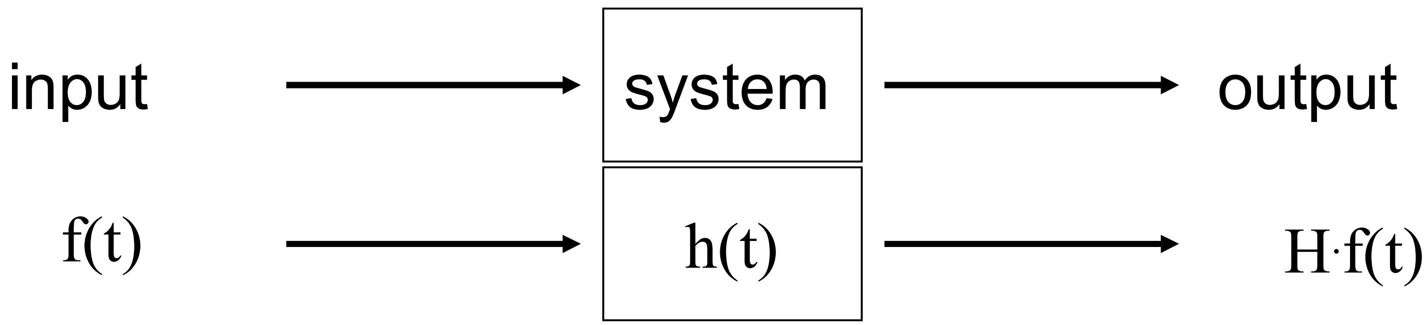
- such functions are called **eigenfunctions**

- example: harmonic functions with constant frequency ω.

$$f(t) = e^{j2\pi\omega t} = \cos(2\pi\omega t) + j\sin(2\pi\omega t)$$

system theory of imaging systems

definitions **Fourier transform**



system response to harmonic function at input:

$$f(t) * h(t) = \int h(\tau) \cdot e^{j2\pi\omega(t-\tau)} d\tau$$

$$= e^{j2\pi\omega t} \cdot \int \underbrace{h(\tau) \cdot e^{-j2\pi\omega\tau} d\tau}_{H} = H \cdot f(t)$$

definitions **Fourier transform**

- the, in general, complex-valued factor H depends on system, frequency, and input function:

$$H(\varpi) = \int h(t) \cdot e^{-j2\pi\varpi t} dt$$

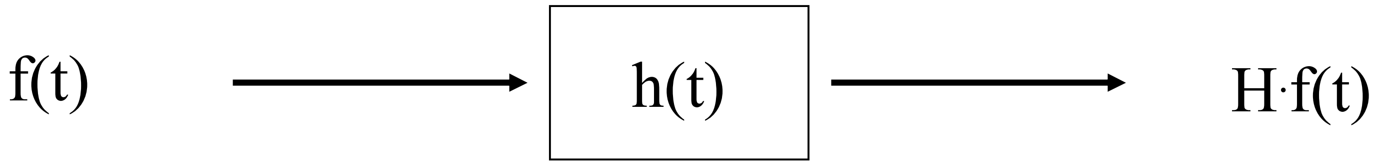
- $H(\omega)$ is called **transfer function** (filter response, frequency response).

- since

$$h(t) = \int H(\varpi) \cdot e^{j2\pi\omega t} d\omega$$

both **impulse response** $h(t)$ and **transfer function** $H(\omega)$ **are equivalent descriptors of a linear stationary systems**

definitions **Fourier transform**



- let $f(t)$ be a **superposition of harmonic functions**.

- the transformation (from time to frequency domain)

$$f(t) = \int F(\omega) \cdot e^{j2\pi\omega t} d\omega$$

and the inverse transformation (from frequency to time domain)

$$F(\omega) = \int f(t) \cdot e^{-j2\pi\omega t} dt$$

is called **Fourier transform**

definitions **Fourier transform**

variant forms of spelling:

$$f(t) \;\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\; F(\omega)$$

$$f(t) \;\circ\!\!-\!\!\!-\!\!\!-\!\!\bullet\; F(\omega)$$

$$F(\omega) = FT\,(f(t))$$

$$f(t) = FT^{-1}\,(F(\omega))$$

definitions **Fourier transform**

properties of the Fourier transform (I):

linearity
$$c_1 \cdot f_1(t) + c_2 \cdot f_2(t) \Leftrightarrow c_1 \cdot F_1(\omega) + c_2 \cdot F_2(\omega)$$

time shift
$$f(t - t_0) \Leftrightarrow F(\omega) \cdot e^{-j2\pi\omega t_0}$$

time/frequency scaling
$$f(a \cdot t) \Leftrightarrow \frac{1}{|a|} \cdot F\left(\frac{\omega}{a}\right)$$

complex conjugate signal
$$f^*(t) \Leftrightarrow F^*(-\omega)$$

time reversal
$$f(-t) \Leftrightarrow F(-\omega)$$

symmetry
$$F(t) \Leftrightarrow f(\omega)$$

definitions **Fourier transform**

properties of the Fourier transform (II):

convolution $$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega) \cdot F_2(\omega)$$

multiplication $$f_1(t) \cdot f_2(t) \Leftrightarrow F_1(\omega) * F_2(\omega)$$

cross-correlation $$f_1(t) \otimes f_2(t) \Leftrightarrow F_1^*(\omega) \cdot F_2(\omega)$$

auto-correlation $$f(t) \otimes f(t) \Leftrightarrow \left|F(\omega)\right|^2$$

integration $$\int_{-\infty}^{+\infty} F(\tau) d\tau \Leftrightarrow \left(j2\pi\omega\right)^{-1} \cdot F(\omega) + \frac{1}{2} F(0)\delta(\omega)$$

differentiation $$\frac{d^n}{dt^n} f(t) \Leftrightarrow \left(j2\pi\omega\right)^n \cdot F(\omega)$$

energy/variance
(Parseval's theorem) $$\int_{-\infty}^{+\infty} \left|f(t)\right|^2 dt = \int_{-\infty}^{+\infty} \left|F(\omega)\right|^2 d\omega$$

definitions **Fourier transform**

properties of the Fourier transform (III):

with system properties *linearity* and *translation invariance* (stationarity), we have:

- the system response is fully characterized by a single function
in time domain: impulse response $h(t)$
in frequency domain: transfer function $H(\omega)$

- equivalent characterization in reciprocal domain (Fourier transform)

$\Rightarrow$
multiplication in given domain $\propto$ convolution in reciprocal domain

definitions **1D Fourier transform**

Fourier transform of time-dependent signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \exp(-j \cdot \omega t) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \exp(+j \cdot \omega t) d\omega$$

FT

FT^{-1}

Fourier transform of location-dependent signals

$$F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-j \cdot 2\pi \cdot ux) dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(u) \cdot \exp(+j \cdot 2\pi \cdot ux) du$$

mapping from spatial domain to frequency domain

$f(x)$ ○———○ $F(u)$

$$F(u) = |F(u)| \cdot \exp(j \cdot \phi(u))$$

$|F(u)|$ = amplitude spectrum

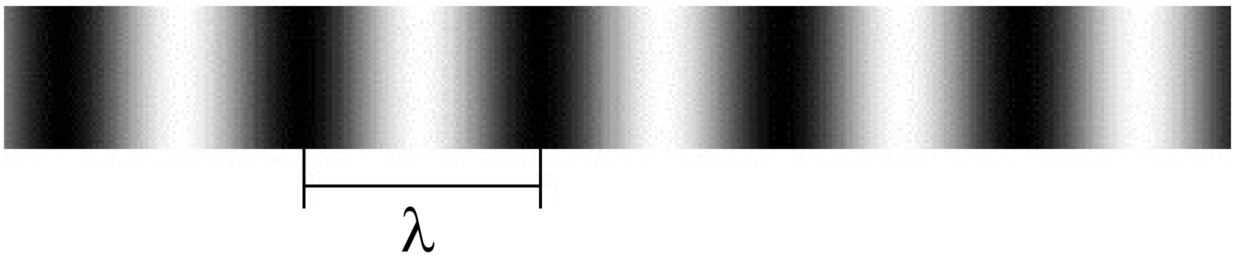
$\phi(u)$ = phase

definitions

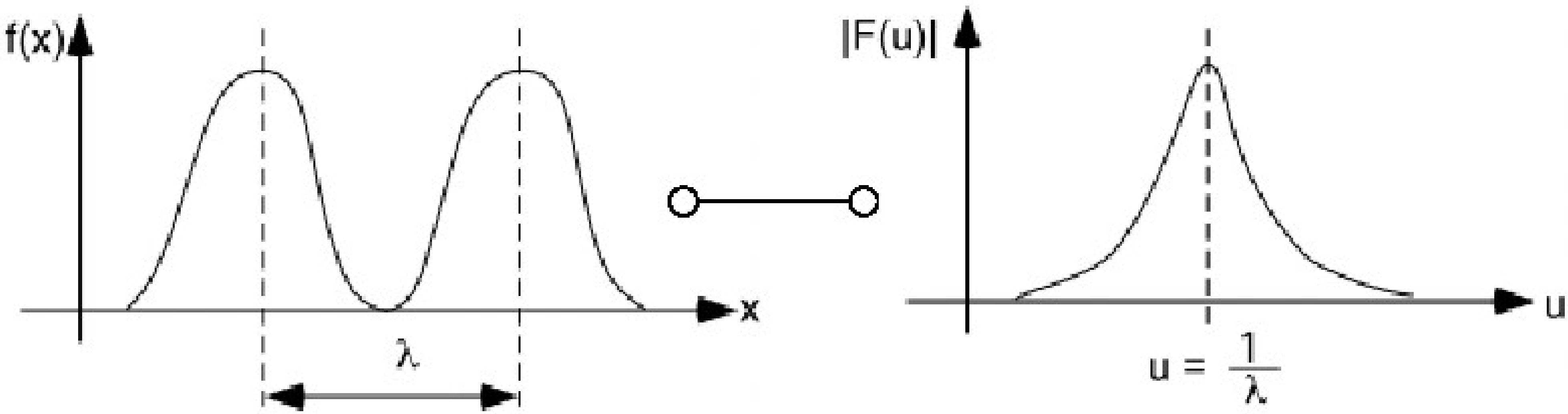
1D Fourier transform

example: sinusoidal signal in spatial domain

λ := wave length in spatial domain
$u=1/\lambda$:= frequency in Fourier domain

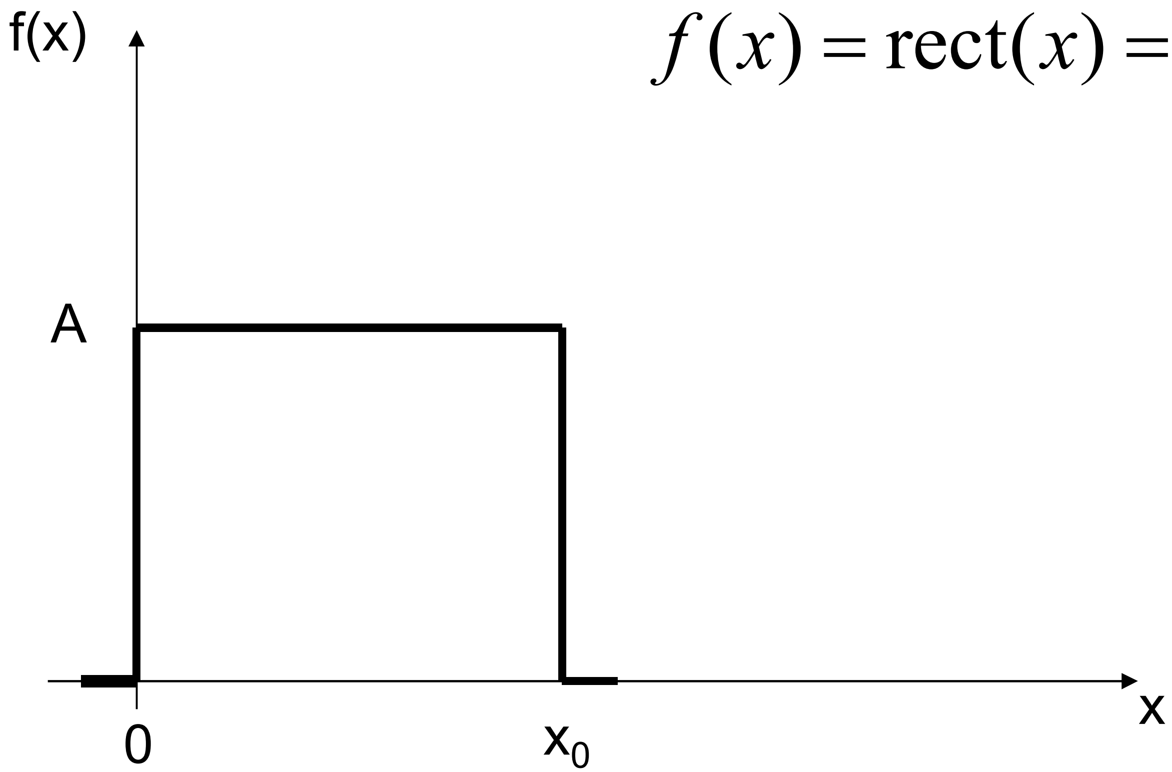


spatial domain amplitude spectrum



definitions **1D Fourier transform**

example: rectangular function in spatial domain



$$f(x) = \mathrm{rect}(x) = \begin{cases} A \text{ if } x \in [0, x_0] \\ 0 \text{ else} \end{cases}$$

definitions **1D Fourier transform**

example: rectangular function in spatial domain

$$F(u)=\int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx=\int_{0}^{x_0} Ae^{-j2\pi ux}dx$$

$$=-\frac{A}{j2\pi u}\left[e^{-j2\pi ux}\right]_0^{x_0}=-\frac{A}{j2\pi u}\left[e^{-j2\pi ux_0}-1\right]$$

$$=\frac{A}{j2\pi u}\left[e^{j\pi ux_0}-e^{-j\pi ux_0}\right]e^{-j\pi ux_0}=\frac{A}{\pi u}\sin(\pi ux_0)e^{-j\pi ux_0}$$

$$\Rightarrow$$

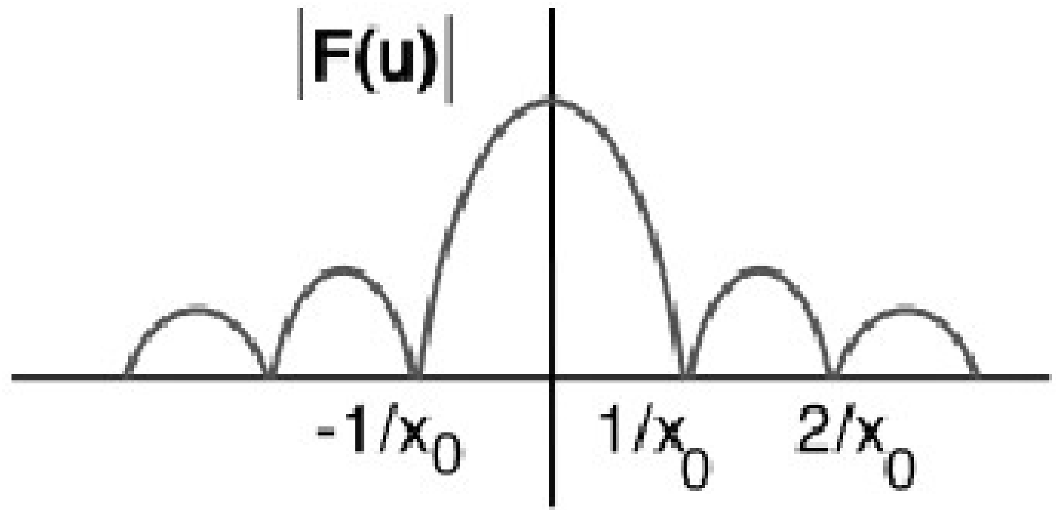
$$\left|F(u)\right|=\frac{A}{\pi u}\left|\sin(\pi ux_0)\right|\left|e^{-j\pi ux_0}\right|=Ax_0\left|\frac{\sin(\pi ux_0)}{\pi ux_0}\right|$$

definitions **1D Fourier transform**

example: rectangular function in spatial domain

amplitude spectrum of rect-function in spatial domain

$$|F(u)| = Ax_0 \left| \frac{\sin(\pi u x_0)}{\pi u x_0} \right|$$



- shifting *f(x)* in spatial domain does not affect *F(u)*
- only phase of *F(u)* is shifted!

definitions **1D Fourier transform**

example: image = matrix consisting of digital grey-values

image: $\left\{\tilde{f}(x_o),\ \tilde{f}(x_o + \Delta x),\ \ldots\ \tilde{f}\left(x_o + (N-1)\cdot\Delta x\right)\right\}$

digitized image: $\left\{f(0),\ f(1),\ \ldots\ f(N-1)\right\} = f(x); \quad x = 0\ \ldots\ N-1$

digital Fourier transform

$$F(u) = \frac{1}{N}\cdot\sum_{x=0}^{N-1} f(x)\cdot\exp(-j\cdot 2\pi\cdot ux/N)$$

$$f(x) = \sum_{u=0}^{N-1} F(u)\cdot\exp(+j\cdot 2\pi\cdot ux/N)$$

digital Fourier transformed : $\left\{F(0),\ F(1),\ \ldots\ F(N-1)\right\} = F(u)$

“true” Fourier transformed: $\left\{\tilde{F}(0),\ \tilde{F}(\Delta u),\ \ldots\ \tilde{F}\left((N-1)\cdot\Delta u\right)\right\}$ $\qquad \Delta u = \frac{1}{N\cdot\Delta x}.$

definitions **1D Fourier transform**

1D-FT and convolution theorem

$$h(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$$

$$\mathrm{FT}\big(h(t)\big) = \mathrm{FT}\big(f(t)\big) \cdot \mathrm{FT}\big(g(t)\big)$$

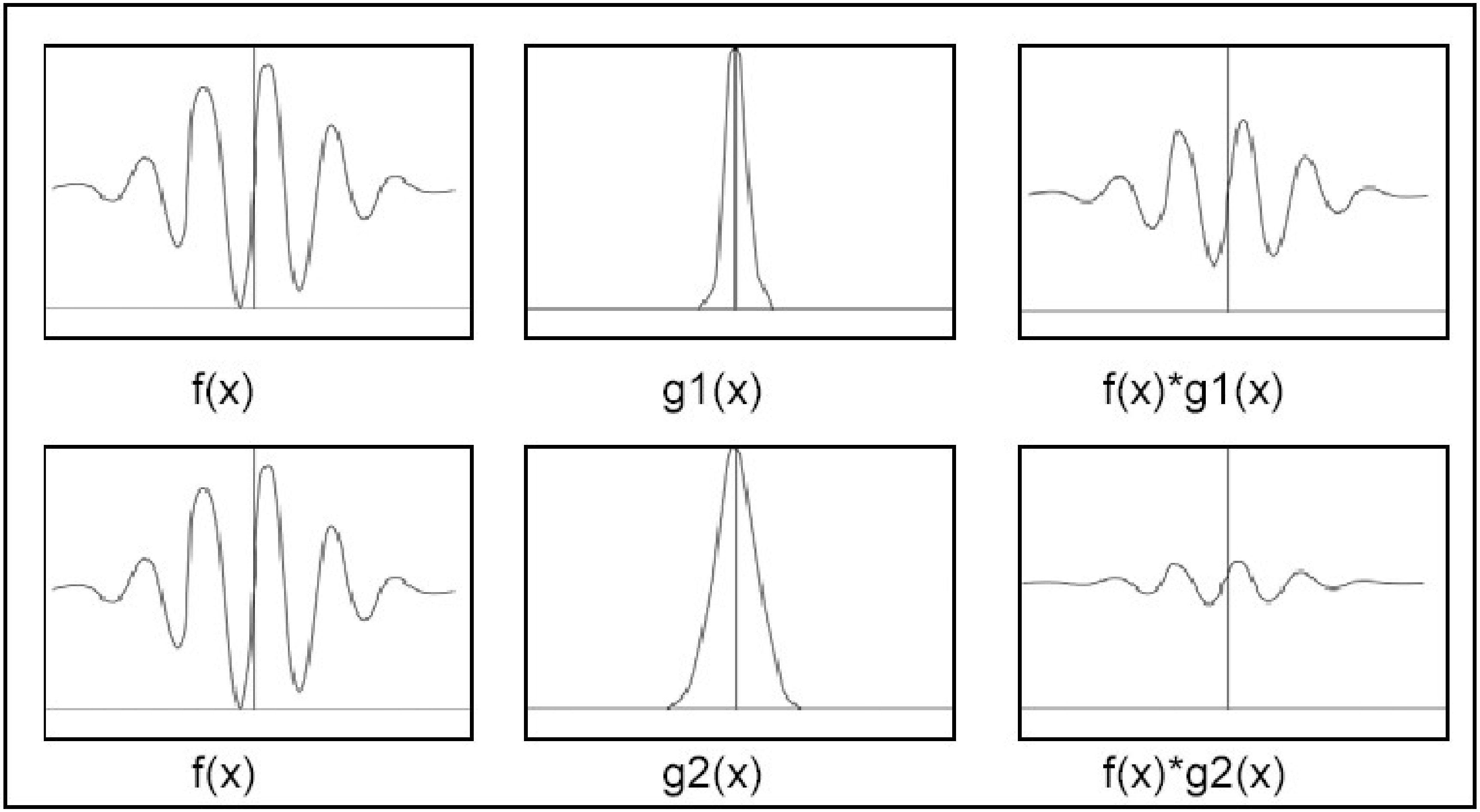
$$\mathrm{FT}^{-1}\big(f(t) \cdot g(t)\big) = \mathrm{FT}^{-1}\big(f(t)\big) * \mathrm{FT}^{-1}\big(g(t)\big)$$

convolution in time domain
corresponds to
multiplication in frequency domain

definitions **1D Fourier transform**

examples:

convolution with: g1: narrow, g2: broad point spread function PSF



definitions

2D Fourier transform

$$F(u,v) = \int_{-\infty}^{+\infty}\int f(x,y)\exp\left(-j \cdot 2\pi \cdot (ux + vy)\right)dxdy$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int F(u,v)\exp\left(+j \cdot 2\pi \cdot (ux + vy)\right)dudv$$

mapping from spatial domain
to frequency domain

spatial domain		frequency domain
$f(x,y)$	o———o	$F(u,v)$

$$F(u,v) = |F(u,v)| \cdot \exp(j \cdot \phi(u,v))$$

$F(u,v) =$ amplitude spectrum
$\phi(u,v) =$ phase

definitions **2D Fourier transform**

2D-FT of quadratic images (N=M=power-of-2 $\rightarrow$ FFT):

$$F(u,v) = \frac{1}{N}\sum_{x,y} f(x,y)\cdot \exp\left(-j\cdot 2\pi(ux+vy)/N\right)$$

$$f(x,y) = \frac{1}{N}\sum_{u,v} F(u,v)\cdot \exp\left(+j\cdot 2\pi(ux+vy)/N\right)$$

if f(x,y) real-valued, then:

$$F(u,v) = F(-u,-v)^*$$

$$F(u+N,v) = F(u,v)$$

$$F(u,v+N) = F(u,v)$$

2D-FT of quadratic images (N=M=power-of-2 $\rightarrow$ FFT):

with Euler's formula, we have:

$$\exp(-j2\pi ux/N) = \cos(-2\pi ux/N) + j\sin(-2\pi ux/N)$$

since cos and sin π-periodic, we find:

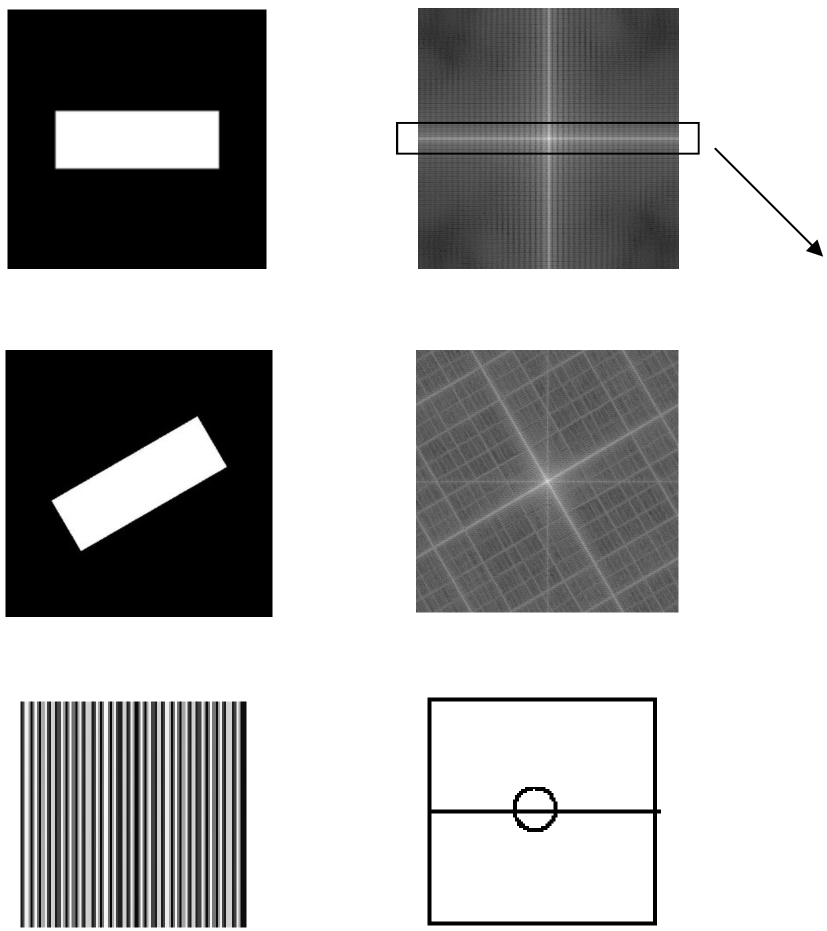
$$F(u+N) = F(u).$$

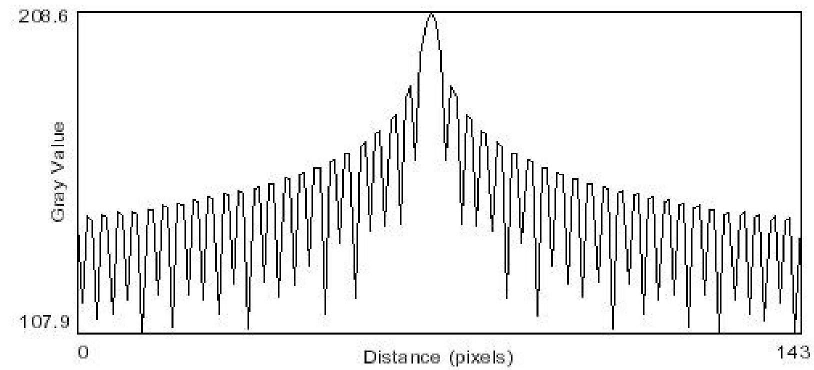
definitions

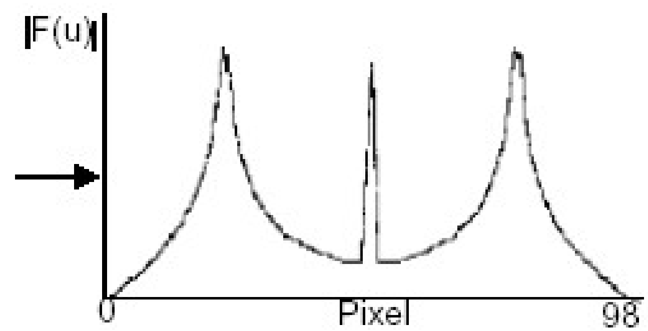
2D Fourier transform

original

amplitude spectrum

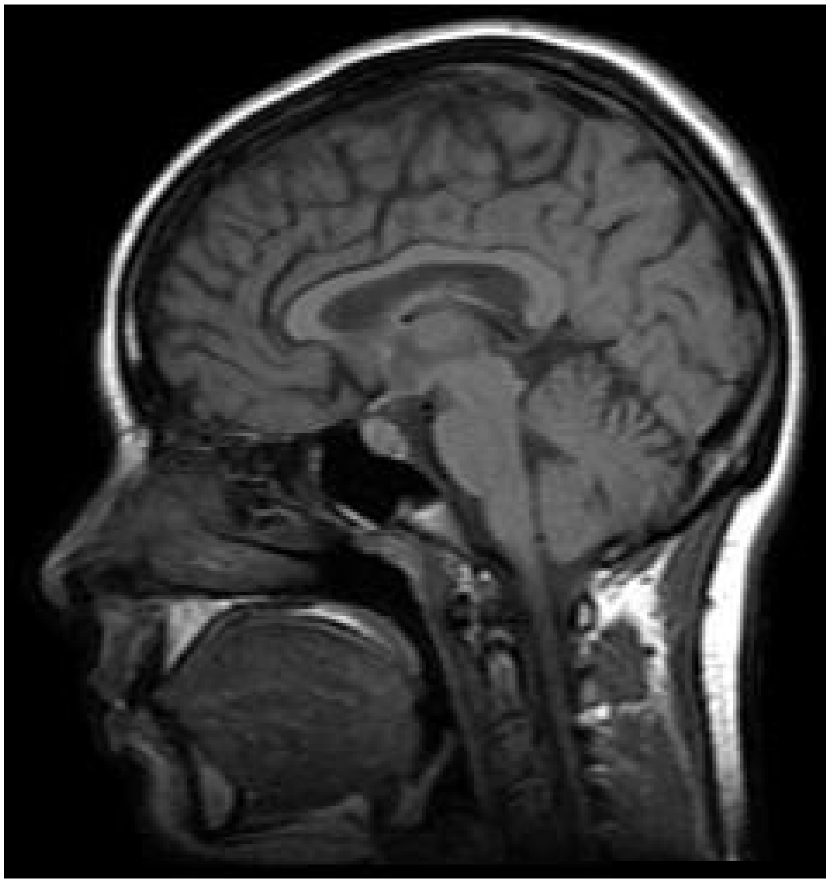




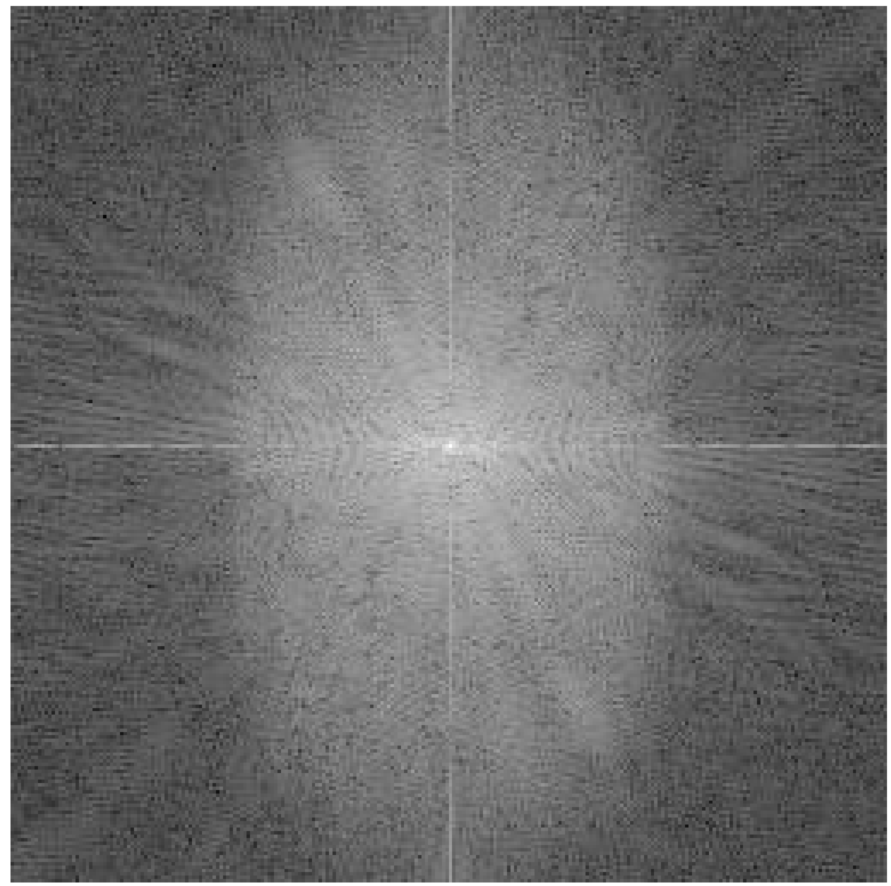


definitions **2D Fourier transform**

original

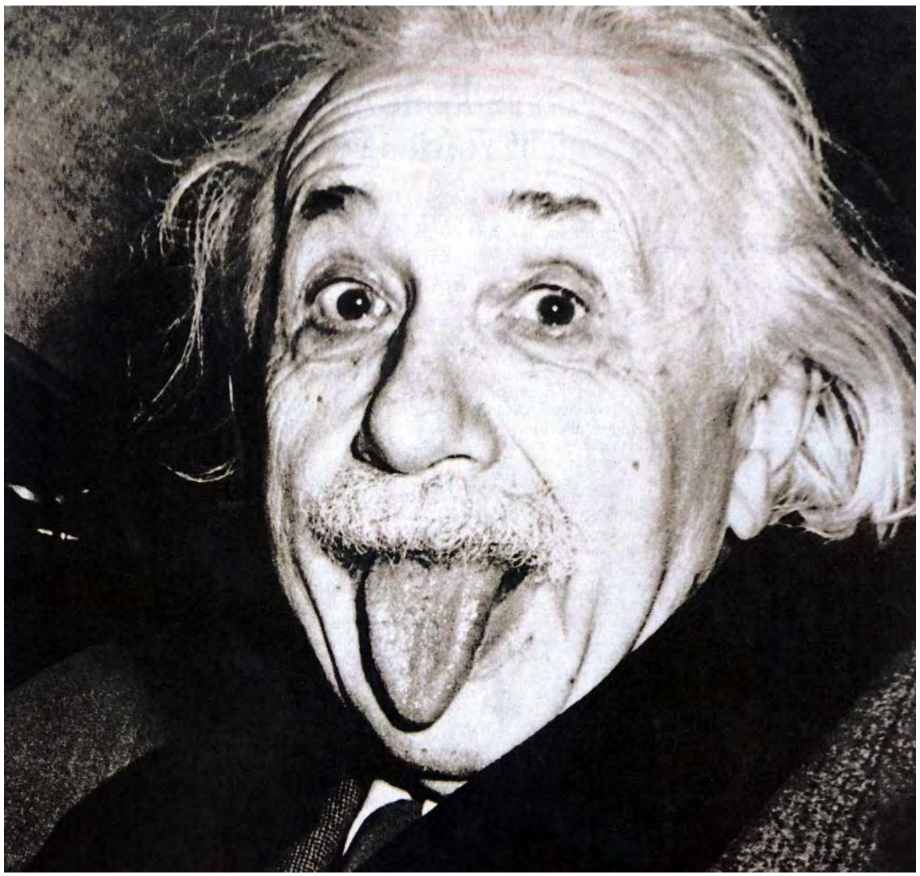


amplitude spectrum

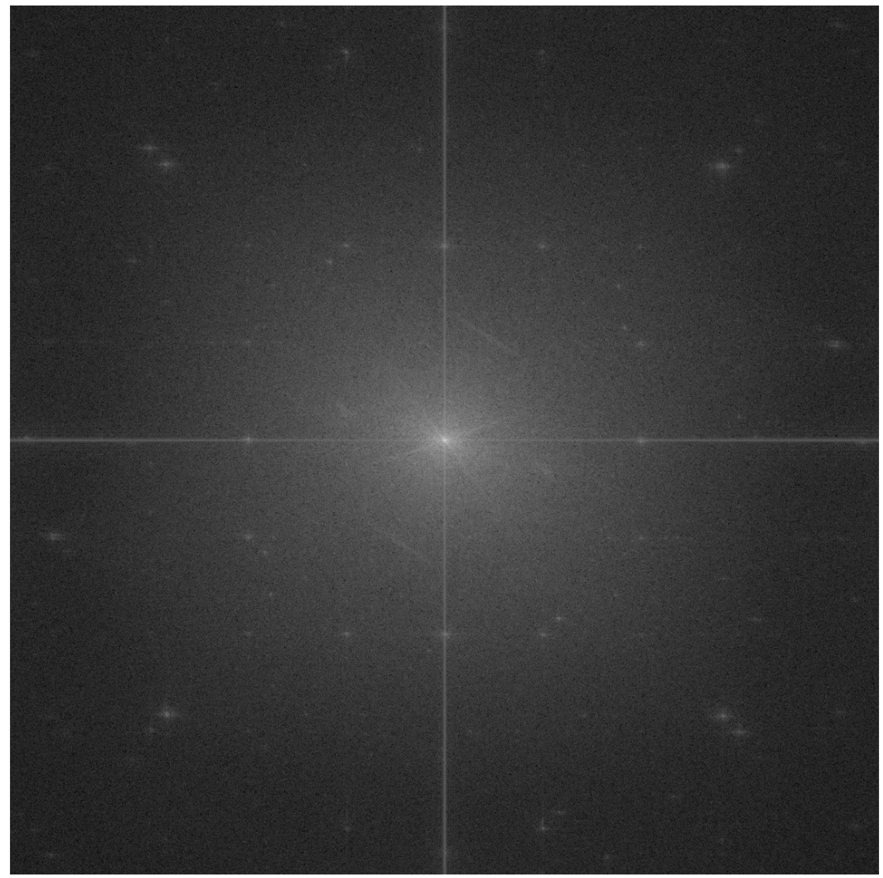


definitions **2D Fourier transform**

original



amplitude spectrum



definitions **2D Fourier transform**

2D-FT and convolution theorem

$$h(x,y) = f(x,y) * g(x,y) = \int_{-\infty}^{+\infty} f(x',y')g(x-x',y-y')dx'dy'$$

$$\text{2DFT}\big(h(x,y)\big) = \text{2DFT}\big(f(x,y)\big) \cdot \text{2DFT}\big(g(x,y)\big)$$

$$\text{2DFT}^{-1}\big(f(x,y) \cdot g(x,y)\big) = \text{2DFT}^{-1}\big(f(x,y)\big) * \text{2DFT}^{-1}\big(g(x,y)\big)$$

definitions **correlation functions**

auto-(cross-)correlation function assesses the correlation of some signal with a delayed copy of itself (or of another signal) as a *function* of delay (time-lag τ)

auto-correlation function

$$C_{vv}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)v(t+\tau)dt$$

$$C_{vv}(-\tau) = C_{vv}(\tau)$$

$$C_{vv}(0) \geq \left|C_{vv}(\tau)\right| \quad \forall\tau$$

$$\tau = lag$$

cross-correlation function

$$C_{vw}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)w(t+\tau)dt$$

$$C_{vw}(-\tau) = C_{wv}(\tau)$$

normalization such that C_{vv} (C_{vw}) = 1 for τ = 0

definitions

correlation functions

1D case:

$$f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(x') \cdot g(x + x') dx'$$

correlation theorem:

$$f(x) \otimes g(x) \quad \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \quad F^*(u) \; G(u)$$

2D case:

$$f(x,y) \otimes g(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x',y') \cdot g(x + x', y + y') dx' dy'$$

correlation theorem:

$$f(x,y) \otimes g(x,y) \quad \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \quad F^*(u,v) \; G(u,v)$$

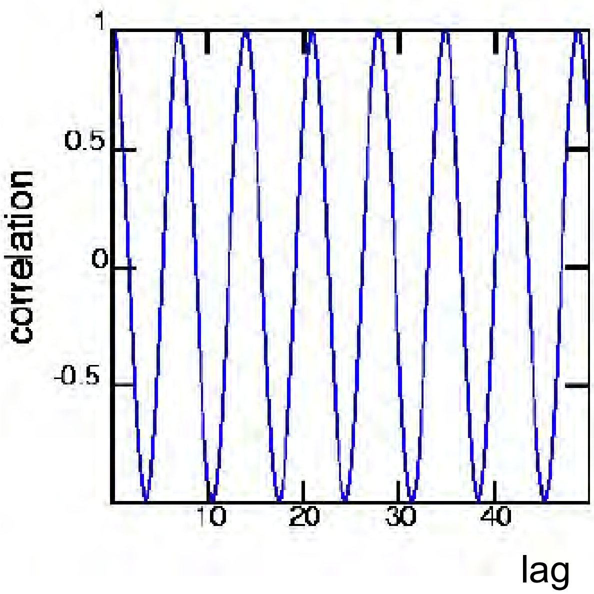
auto-correlation function:
(Wiener-Khinchin theorem)

$$f(x,y) \otimes f(x,y) \quad \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \quad |F(u,v)|^2$$

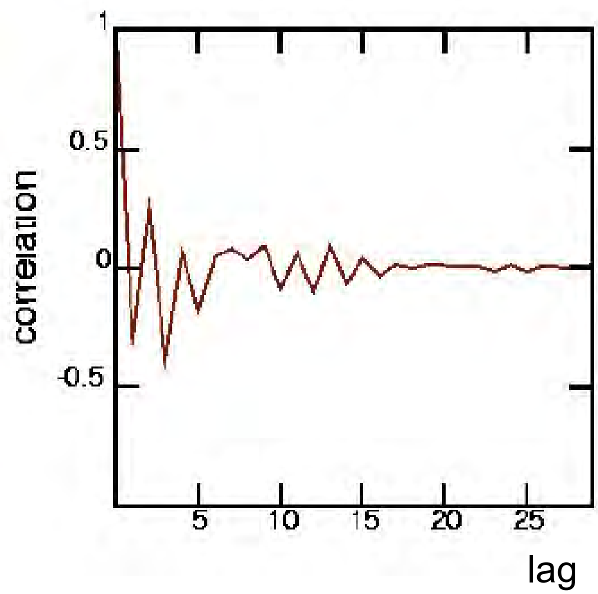
definitions **correlation functions**

example:
autocorrelation functions of some signals

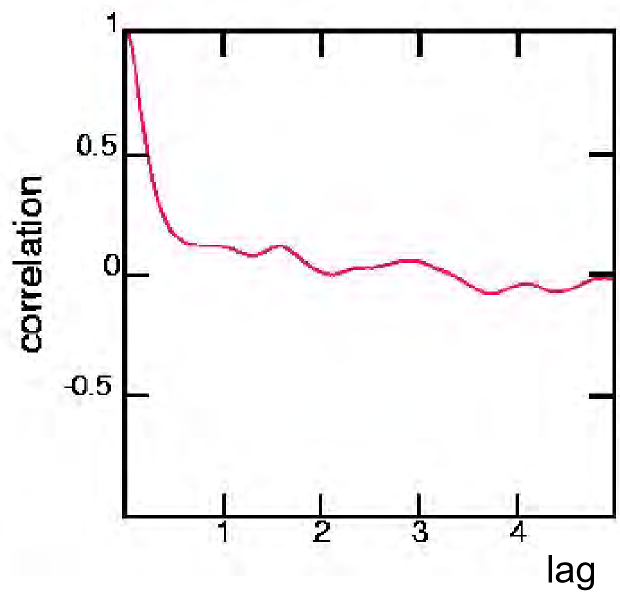
periodic signal



stochastic signal

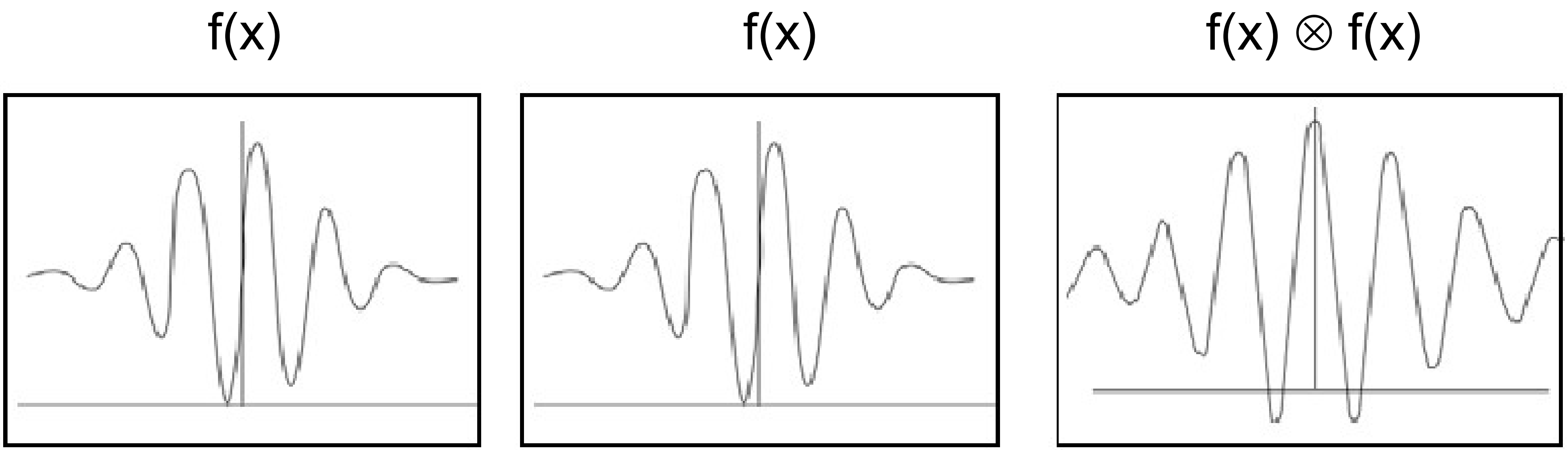


signal with "memory"



definitions **correlation functions**

example:
at which shifts has function f(x) highest similarity with itself ?

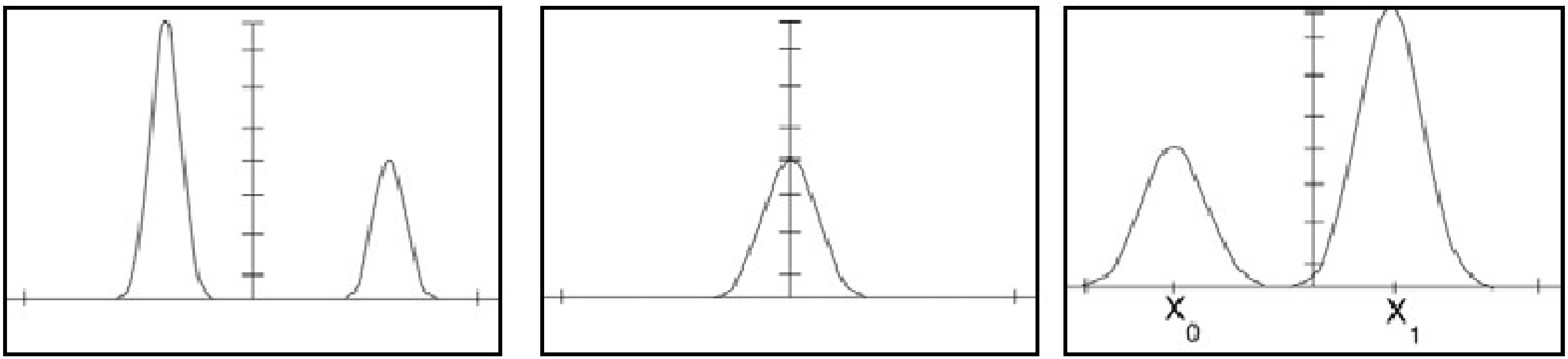


the auto-correlation function f(x) ⊗ f(x) exhibits several maxima

definitions **correlation functions**

example:
at which shifts has function g(x) highest similarity with function f(x) ?

f(x) g(x) $f(x) \otimes g(x)$



the cross-correlation function $f(x) \otimes g(x)$ exhibits two maxima

main theorem of system theory of imaging systems:

for a linear and translation-invariant system, there exists a function $h(x,y)$, such that:

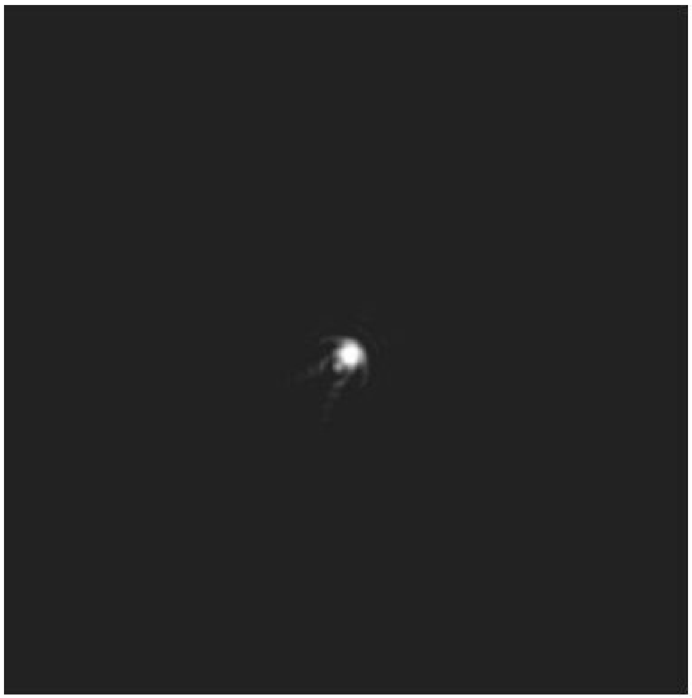
$$g(x,y) = f(x,y) * h(x,y) = \int_{-\infty}^{+\infty}\int f(x',y')h(x-x',y-y')dx'dy'$$

imaging systems: $h(x,y)$ is called *point spread function* (PSF)

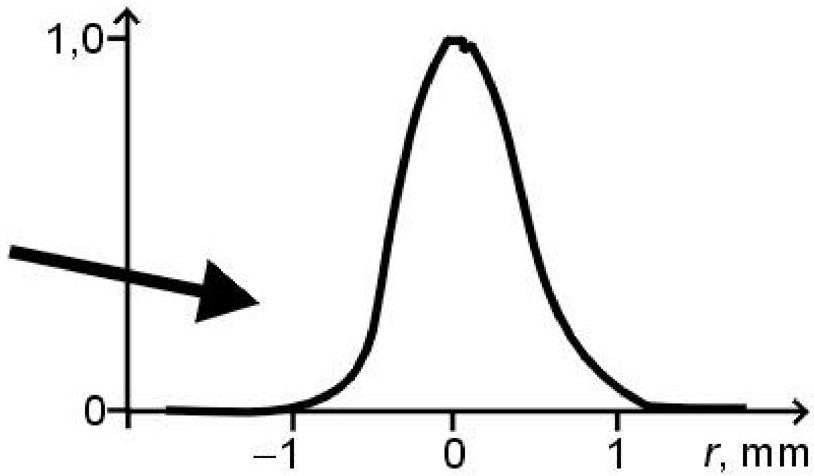
signal processing: $h(t)$ is called *impulse response function:*

$$g(t) = f(t) * h(t) = \int_{-\infty}^{+\infty} f(t)h(t-\tau)d\tau$$

point spread function



point spread function



CT-image of
wire
(phantom measurement)

main theorem of system theory of imaging systems:

with convolution theorem, we have:

$$g(x,y) = f(x,y) * h(x,y) \quad \circ\!\!-\!\!-\!\!\circ \quad G(u,v) = F(u,v) \cdot H(u,v)$$

$$h(x,y) \quad \circ\!\!-\!\!-\!\!\circ \quad H(u,v)$$

the function H(u,v) is called *transfer function* and is the Fourier transform of the point spread function h(x,y)

signal processing: $H(\omega)$ **is called filter (of frequency) *response***

definitions **modulation transfer function**

general definition of MTF:

MTF = modulation transfer function

$$\mathrm{MTF}(u,v) = |H(u,v)|$$

and more exactly:

$$\mathrm{MTF}(u,v) = \frac{|H(u,v)|}{|H(0,0)|}$$

MTF = absolute value of transfer function normalized to 1 at the origin

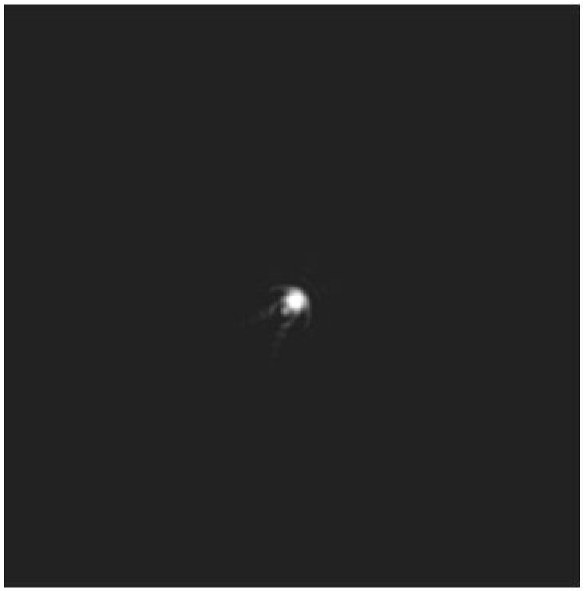
modulation transfer function for sinusoidal functions

$$\mathrm{MTF}(u,v) = \frac{|H(u,v)|}{|H(0,0)|}$$

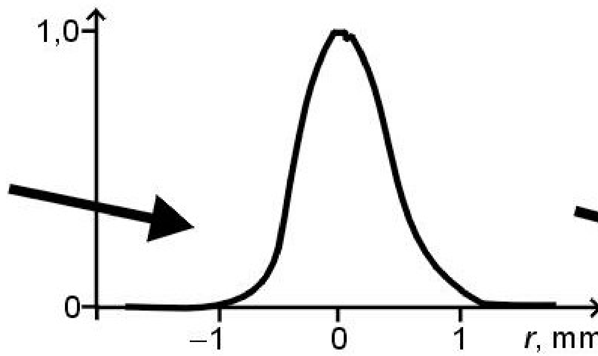
$$= \frac{\text{amplitude @output}}{\text{amplitude @input}}$$

$$= \frac{\text{contrast @output}}{\text{contrast @input}}$$

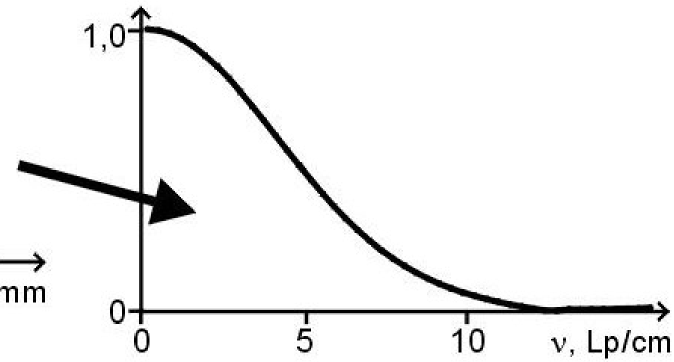
modulation transfer function MTF



CT-image of wire

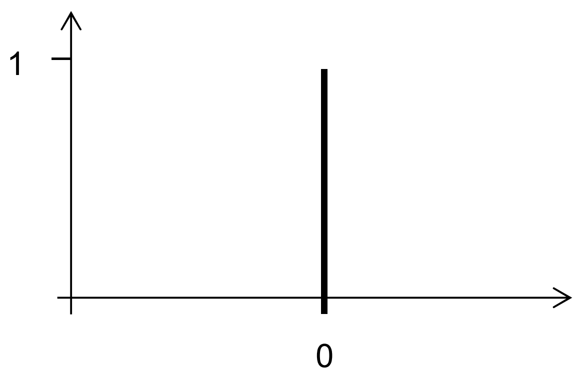


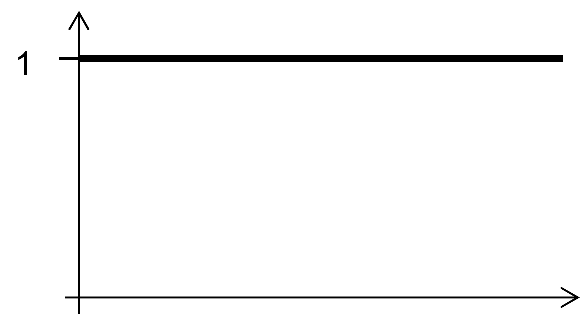
point spread function



modulation transfer function

ideal system





definitions **noise**

noise → system → output

- let r(x,y) denote a noise input image (quantum noise)

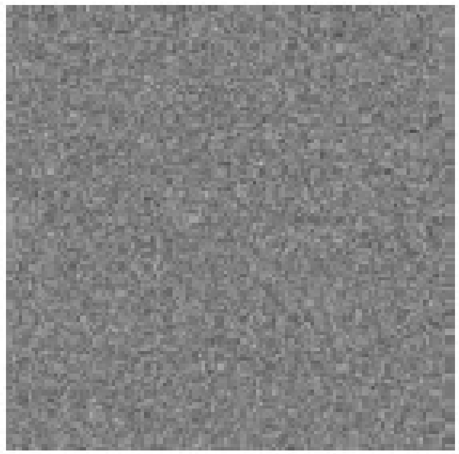
- the image does not contain any information:
  Fourier spectrum is white, there are no correlations between pixels

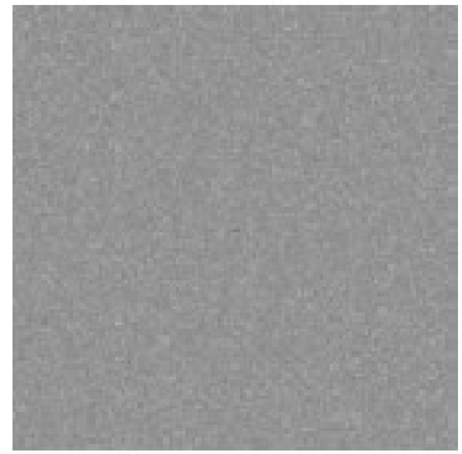
- let R(u,v) denote the 2D-Fourier transform of r(x,y)
  (noise amplitude spectrum).

$$r(x,y) \circ\!\!-\!\!\!-\!\!\circ R(u,v)$$

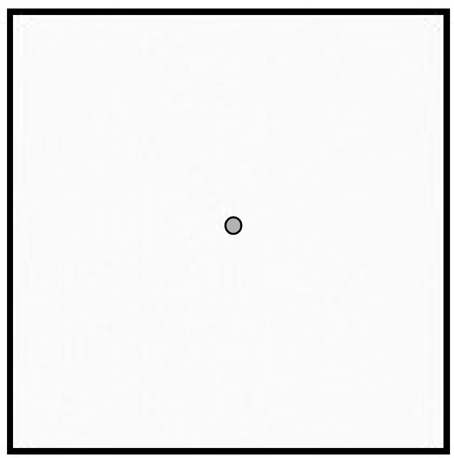
$$|R(u,v)^2| = NPS_{input} = \text{noise power spectrum}$$

definitions **noise**


noise @input


amplitude spectrum

white noise:
2D autocorrelation
function $\neq 0$ at origin
only


autocorrelation fct

definitions **noise**

- let r(x,y) pass through ideal imaging system
- system is noiseless (does not add noise to output)
- noise @output can be explained by quantum noise only !!
  (i.e., Detective Quantum Efficiency DQE=1)

- for quantum noise, we have: $$\text{DQE} = \frac{\text{mean number of detected } \gamma\text{-quanta}}{\text{mean number of incoming } \gamma\text{-quanta}}$$

- with Fourier transform and transfer function

image @output		Fourier transform @output
$r(x,y) * h(x,y)$	○—●	$R(u,v) \cdot H(u,v)$

the noise power spectrum @output reads (*Wiener spectrum* W(u,v)):

$$NPS_{output}(u,v) = W(u,v) = |R(u,v)|^2 \cdot |H(u,v)|^2 = |R(u,v)|^2 \cdot MTF(u,v)^2$$

(DQE = 1!)

definitions **noise**

- noise power spectrum @output:

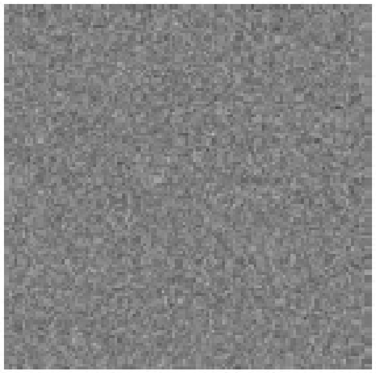
$$NPS_{output}(u,v) = k \cdot MTF(u,v)^2$$

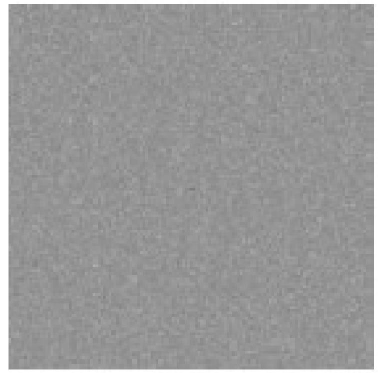
k = proportionality factor
not dependent on noise power spectrum @input

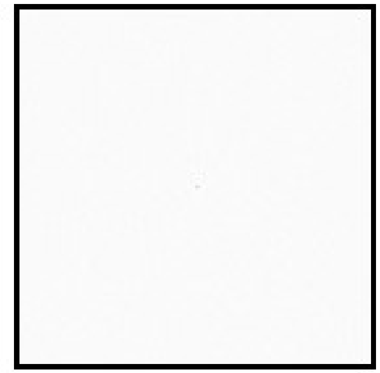
- **noise power spectrum @output and squared MTF have same functional form!**

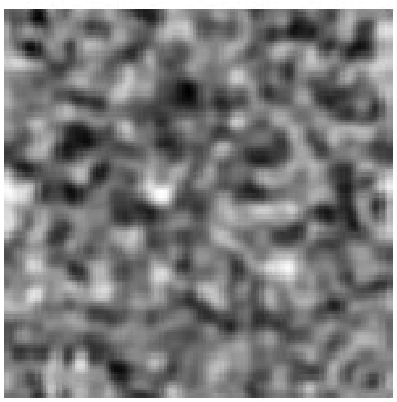
- **since FT (autocorrelation function) = noise power spectrum**
  ⇒ autocorrelation function @output and squared MTF have same functional form!

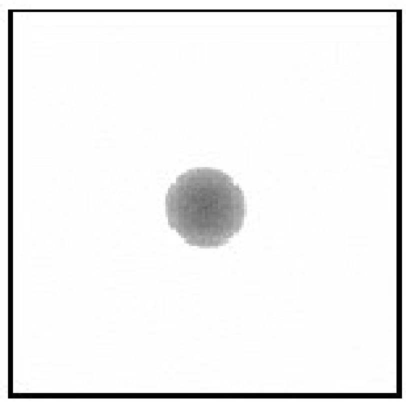
definitions **noise**

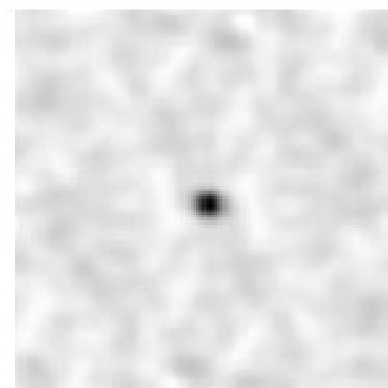

noise @input


amplitude spectrum

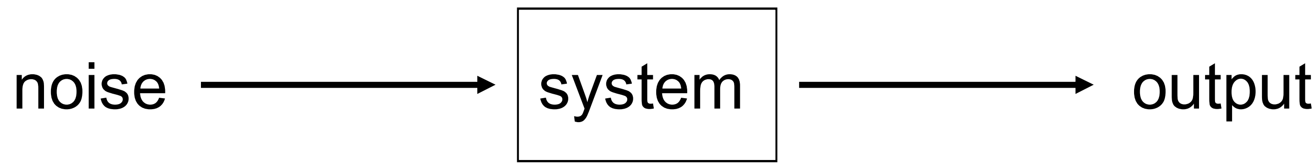

autocorrelation fct


noise @output


amplitude spectrum


autocorrelation fct

definitions **noise**



- neighboring pixel in an output image of an imaging system are no longer independent from each other

- a finite MTF truncates higher spatial frequencies

- a band-limited Fourier spectrum is equivalent to stronger correlations in an image

- an imaging system with finite MTF generates correlations in the output image

definitions **noise**

- Detective Quantum Efficiency (DQE): a measure for image quality

$$\text{DQE} = \frac{(\text{signal/noise ratio})^2 \text{ @output}}{(\text{signal/noise ratio})^2 \text{ @input}}$$

- factor, by which the system deteriorates the signal/noise ratio

- if only noise@input:
  factor, by which the system deteriorates noise

- previous assumption: system does not add noise to output
  (ideal system; DQE = 1)

definitions **noise**

- signal @output can be estimated from signal @input using the transfer function H(u,v) :

$$\text{signal}_{\text{output}}(u,v) = \text{signal}_{\text{input}}(u,v) \cdot H(u,v)$$

- with MTF(u,v) = |H(u,v)|, we have

$$\text{signal}^2_{\text{output}}(u,v) = G^2 \cdot \text{signal}^2_{\text{input}}(u,v) \cdot \text{MTF}^2(u,v)$$

where G = amplification factor (gain); system-dependent

⇒ **generalized DQE**

$$\text{DQE}(u,v) = G^2 \cdot \text{MTF}^2(u,v) \cdot \frac{\text{NPS}_{\text{input}}(u,v)}{\text{NPS}_{\text{output}}(u,v)}$$

DQE and MTF for an imaging system

with: $$DQE(u,v) = G^2 \cdot MTF^2(u,v) \cdot \frac{NPS_{input}(u,v)}{NPS_{output}(u,v)}$$

we have: $$NPS_{output}(u,v) = G^2 \cdot \frac{MTF^2(u,v)}{DQE(u,v)} \cdot NPS_{input}(u,v)$$

even if DQE(u,v) = const. for high spatial frequencies (u,v),
the noise power spectrum is reduced by the MTF(u,v)

if DQE(u,v) decreases more rapidly to 0 than MTF(u,v),
the noise power spectrum in this frequency band is strongly enhanced

DQE and MTF for an imaging system

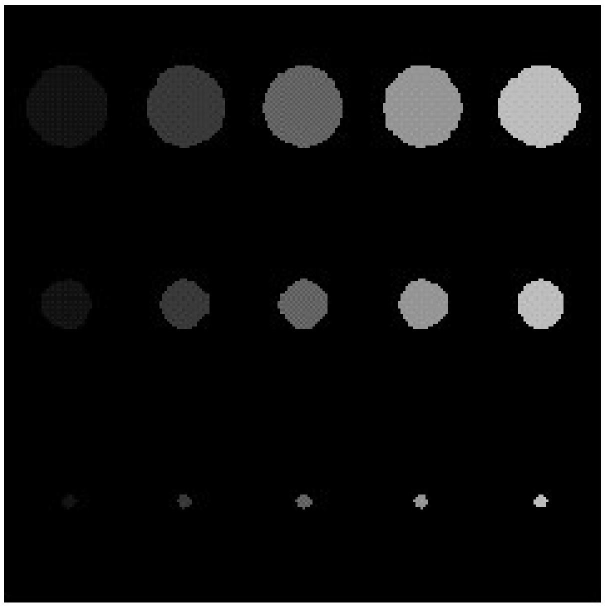
- MTF is always finite in real imaging systems
  band limitation, correlations
  diminished spatial resolution

- there is always noise in real imaging systems
  DQE < 1

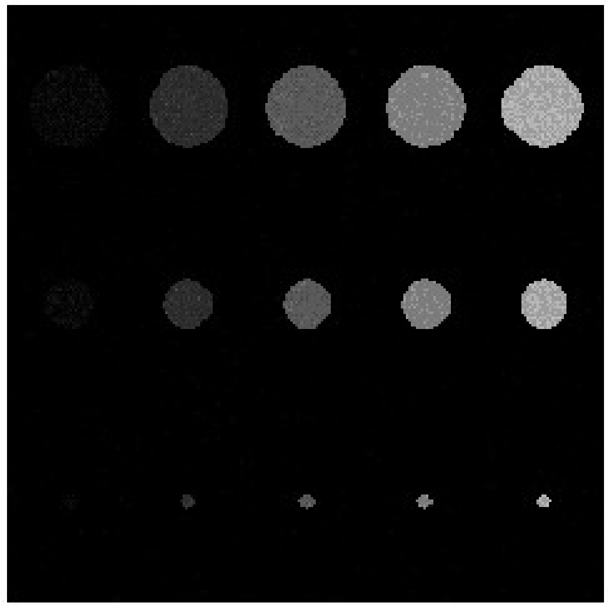
improving DQE results in a deteriorated MTF and vice versa !

definitions **noise**

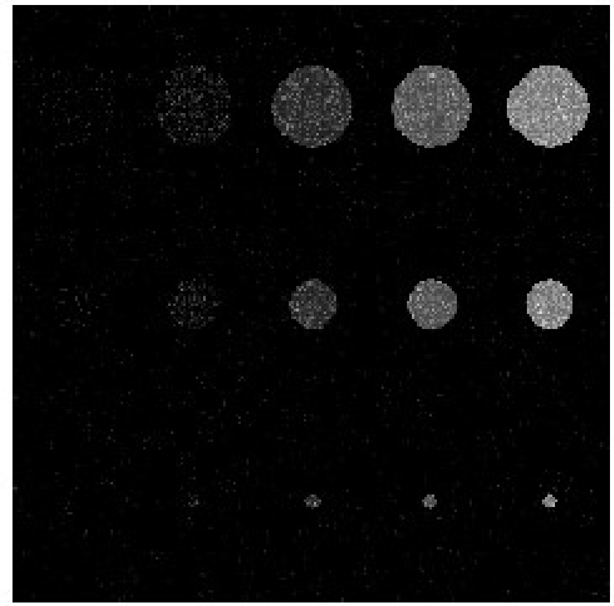
DQE and MTF for an imaging system



no noise



256 quanta/pixel
noise +/- 16



16 quanta/pixel
noise +/- 4

definitions **sampling**

digitization:
conversion of continuous (amplitudes) grey-scale values into digital discrete (amplitudes) grey-level values

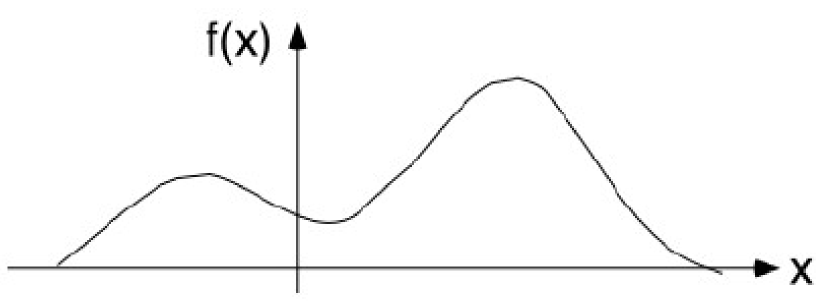
quantization:
conversion of an analogue (signal) image into discrete (values) pixel

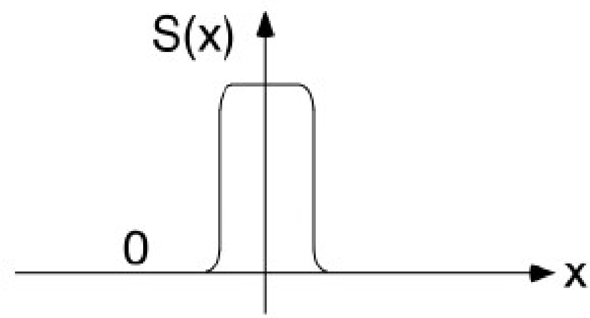
quantization error:
example: 10 bit ADC
range: 0 - 1024
single value: $q = \frac{1024}{2^{10}} = 1$

quantization error (q/2) = 0,5

definitions **sampling**

given image f(x,y) and sensors with sensitivity curve S(x,y)



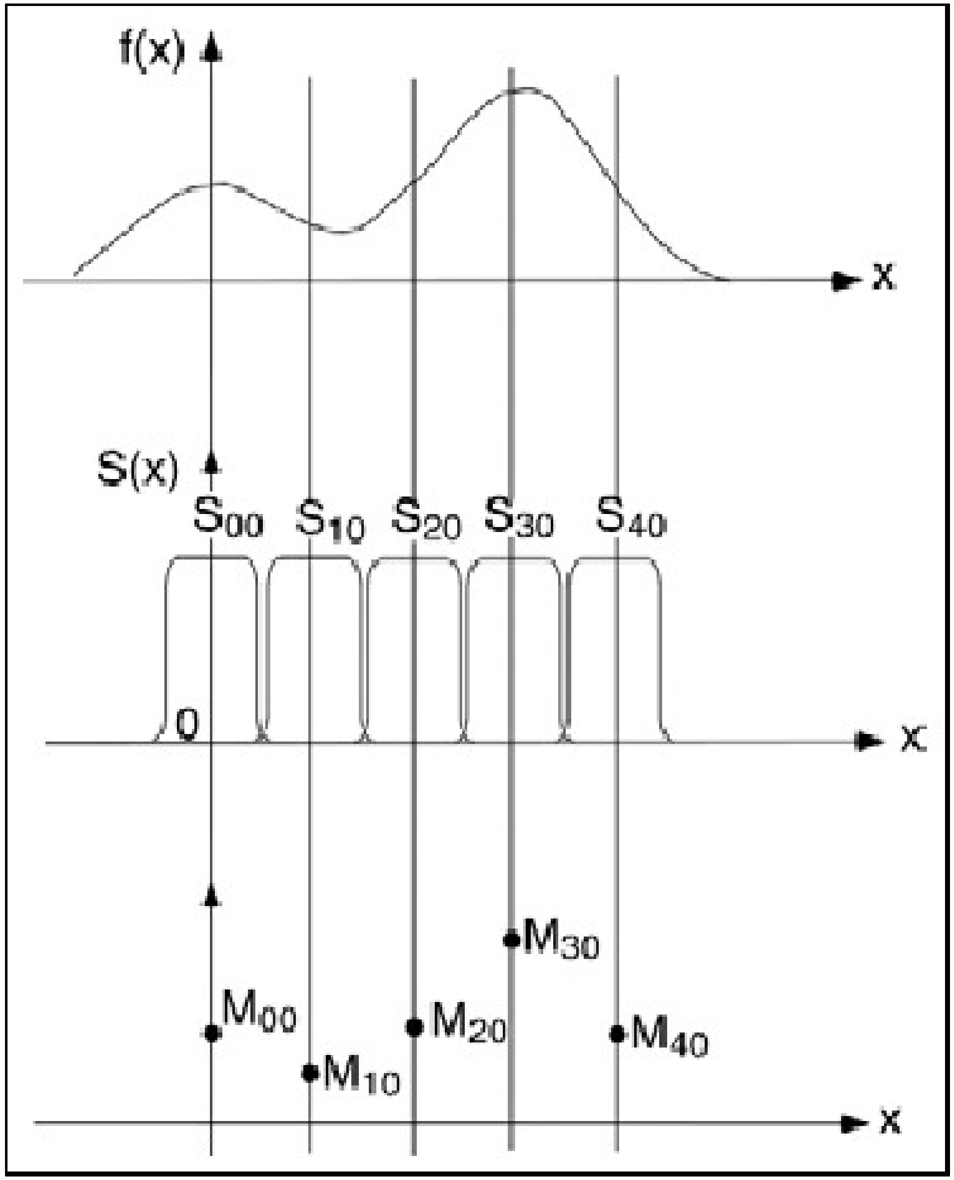


signal from sensor (n,m):

$$M_{nm} = \iint f(x,y) \cdot S(x - n \cdot \Delta x, y - m \cdot \Delta y) dxdy$$

mathematically: S(x,y) is 2D-Dirac function
multiplication (convolution) of image with a **comb-like function**, that attains the value 1 in the center of a pixel and the value 0 otherwise

definitions **sampling**



definitions **sampling theorem**

signal processing:

sampling interval and Nyquist frequency

$$v_n = \sum_{n=-\infty}^{\infty} v(t)\delta(t - n\Delta t)$$

$$v_n = v(n\Delta t); \quad n = ..., -3, -2, -1, 0, 1, 2, 3, ...$$

Δt is called sampling interval

$$\omega_{Nyquist} \equiv \frac{1}{2\Delta t}$$

definitions **sampling theorem**

Let $v(\mathrm{t})$ denote a continuous and band-limited function, sampled with sampling interval $\Delta \mathrm{t}$:

$$V(\omega) = 0 \quad \forall |\omega| > \omega_{Nyquist}$$

where $V(\omega)$ denotes the Fourier spectrum of $v(\mathrm{t})$.

$v(\mathrm{t})$ is then fully determined by the sampling values v_n :

$$v(t) = \Delta t \sum_{n=-\infty}^{+\infty} v_n \frac{\sin(2\pi\omega_{Nyquist}(t - n\Delta t))}{\pi(t - n\Delta t)} \propto v(t) * \mathrm{sinc}\left(\frac{t}{\Delta t}\right)$$

definitions **sampling theorem**

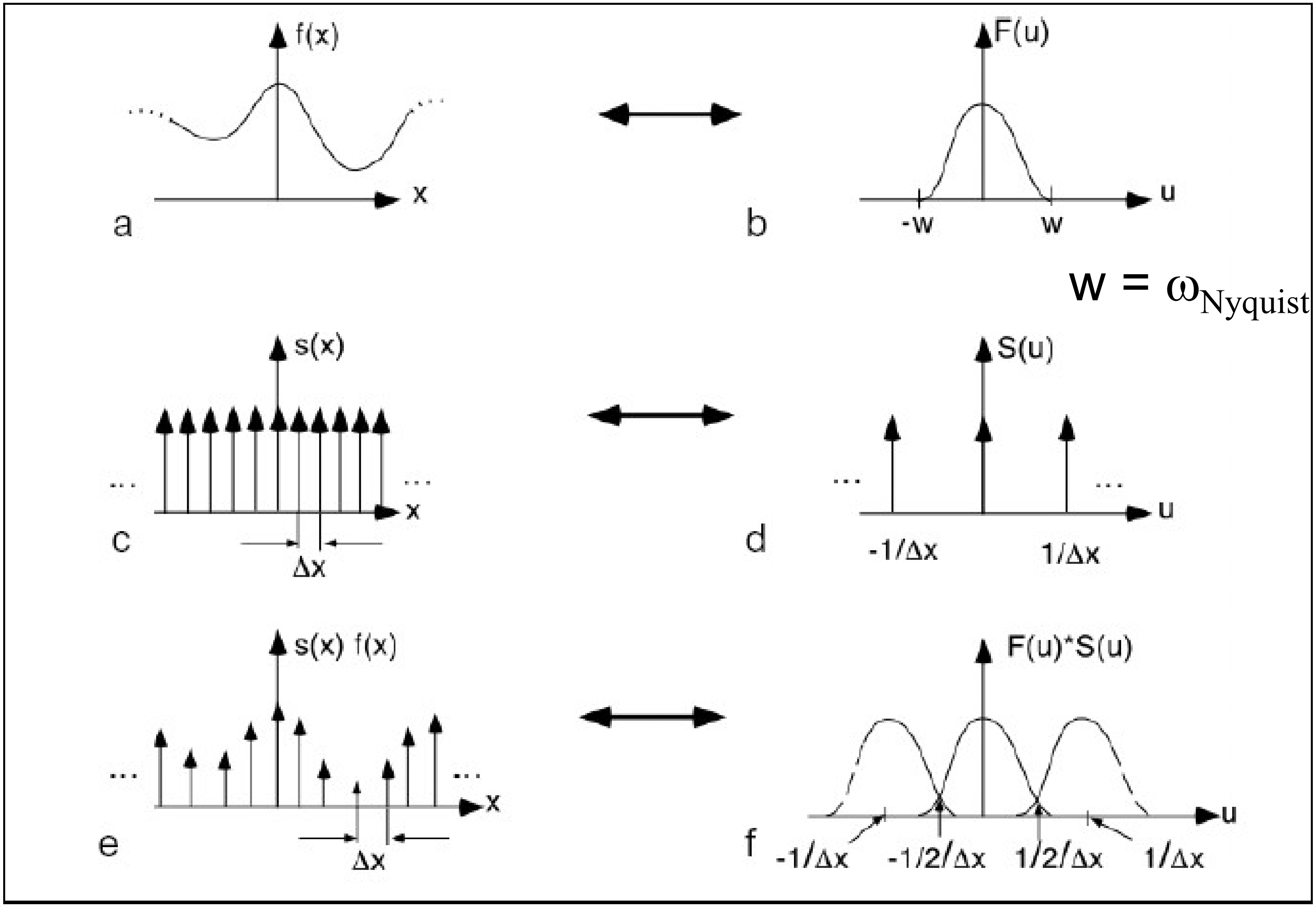
image processing:
sampling interval and Nyquist frequency

Δx is called spatial sampling interval

$$\omega_{Nyquist} \equiv \frac{1}{2\Delta x}$$

$\omega_{Nyquist}$ is the highest spatial frequency

definitions **sampling theorem**



$w = \omega_{Nyquist}$

definitions **aliasing**

- aliasing: sampling a ***non-band-limited*** continuous function

$$V(\omega) \neq 0 \quad |\omega| > \omega_{Nyquist}$$

- these spectral components are (somehow) convolved to the interval

$$|\omega| \leq \omega_{Nyquist}$$

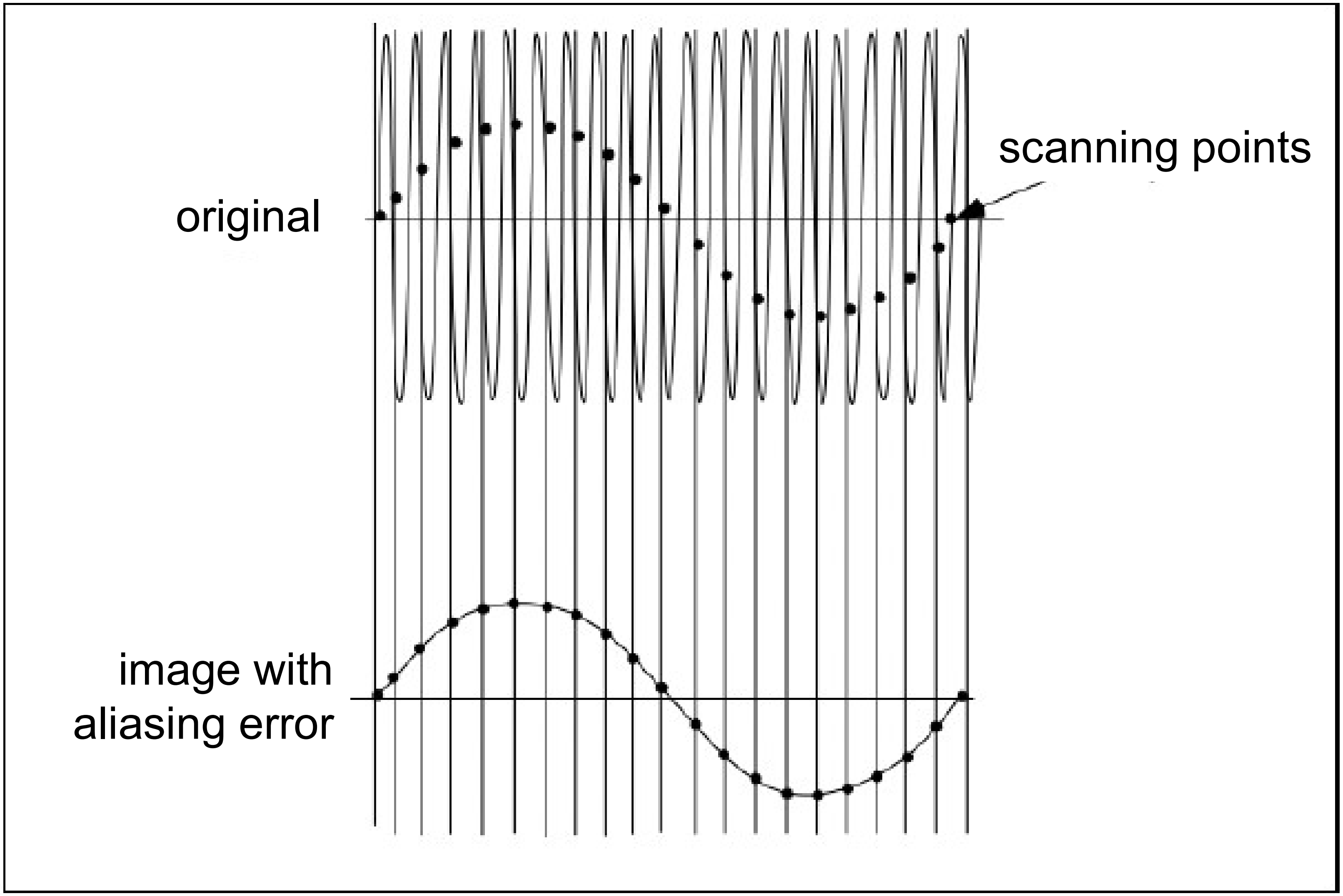
solution:

(1) bandwidth of signal known *a priori* or limited prior to sampling using some filter

(2) adequate sampling

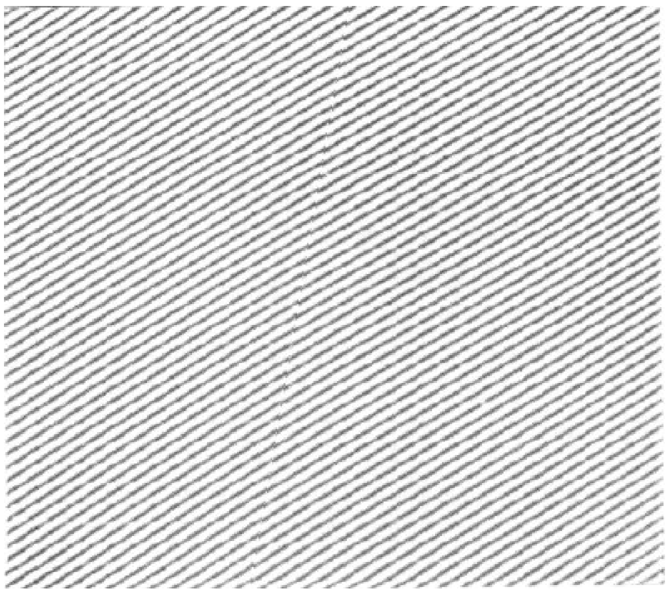
definitions

aliasing

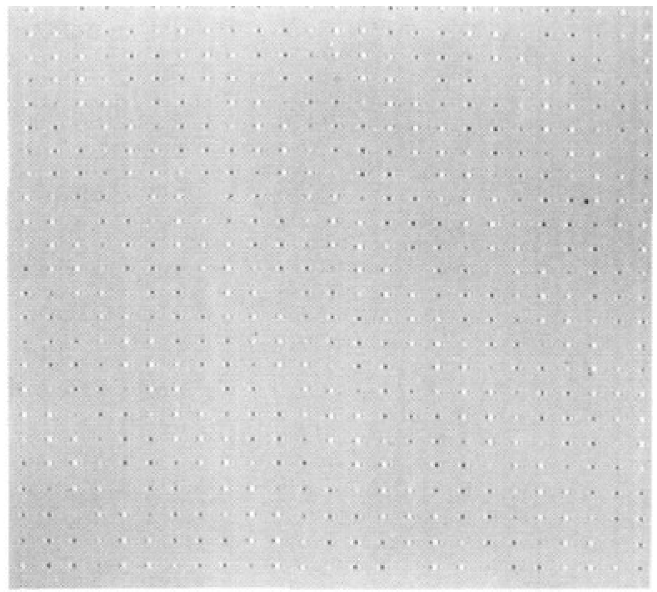


definitions **aliasing**

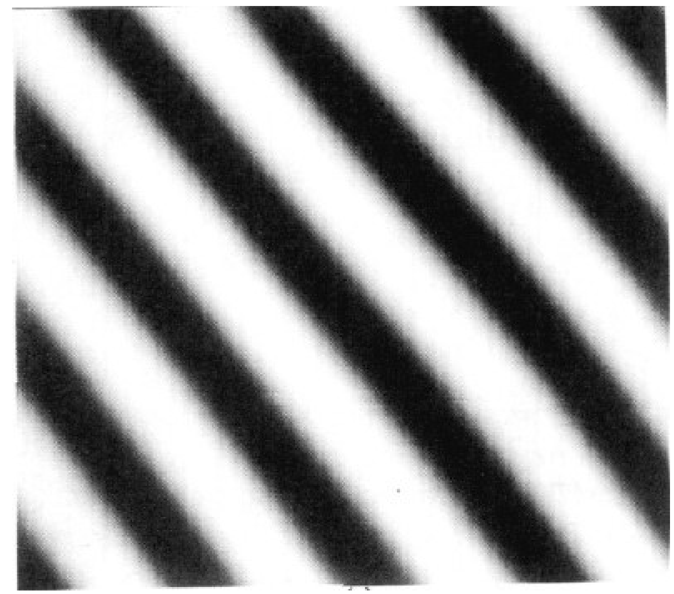
Moiré effect (change of orientation and frequency)



high-frequent sinusoidal input image



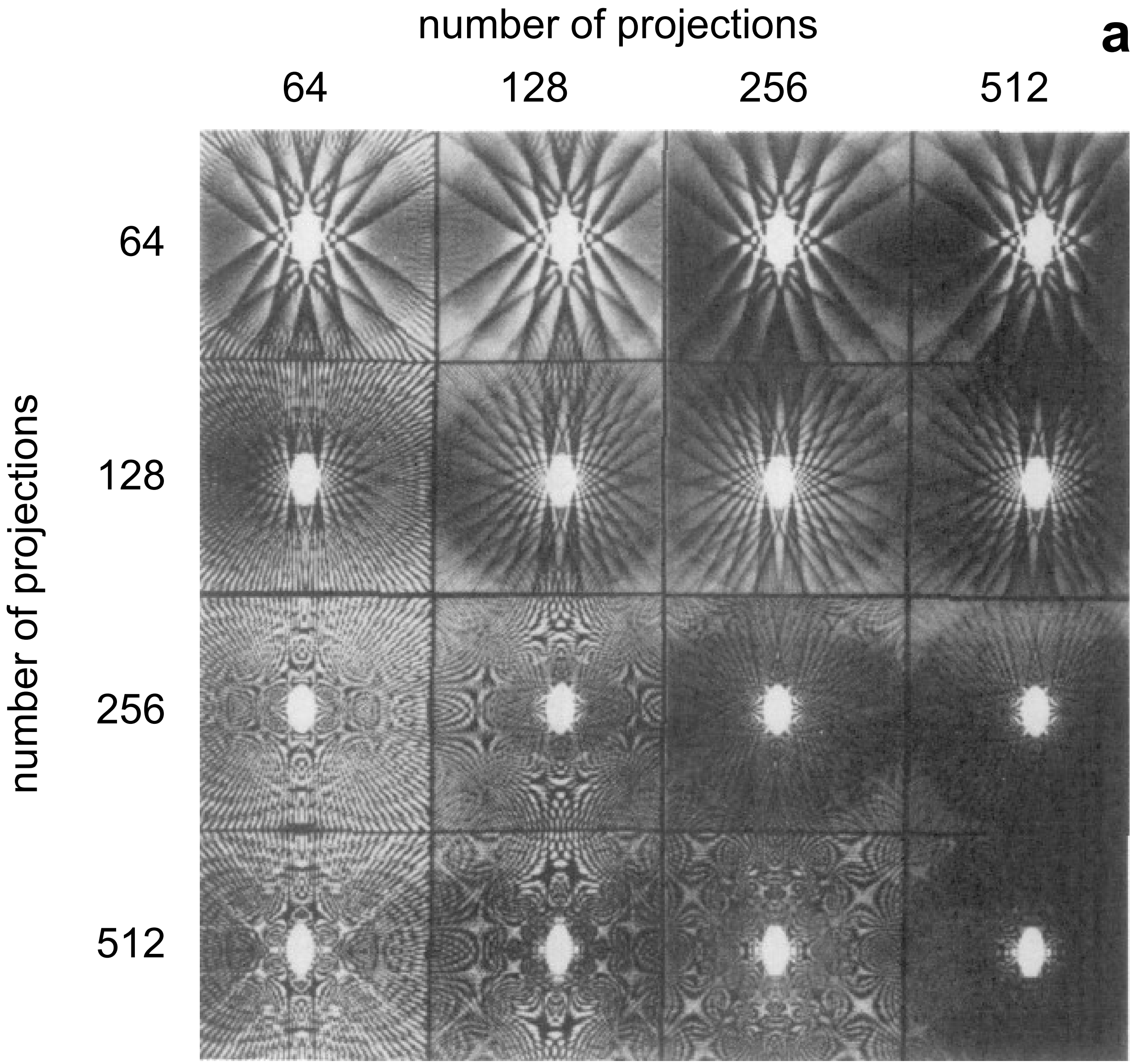
inadequately sampled image $\Delta x << \omega_{Nyquist}$



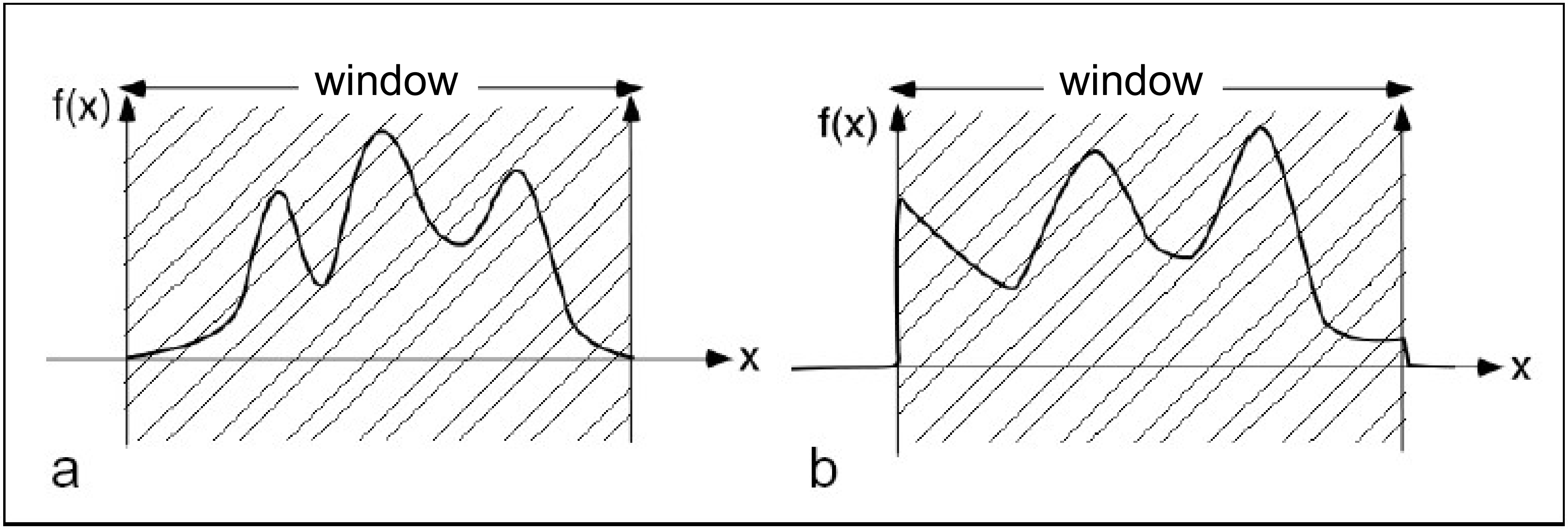
lowpass-filtered image $F_{cutoff} = \omega_{Nyquist}$

definitions **aliasing**

inadequate sampling of an ellipse



definitions **band limiting**



function in (b) not band-limited:
strong edges = Dirac functions = *white* Fourier spectrum
requires ***tapering*** prior to digitization
(multiplication with suitable “window function” (other than boxcar))

definitions **sampling**

sampling: multiplication of signal (image) with comb-like function

periodisation: convolution of signal with comb-like function
(Nyquist condition: periodic continuation)

sampling theorem: when following the Nyquist condition, the band-limited interpolation (sinc-series) yields the original function, or leads to aliasing errors otherwise

sampling frequency: at least twice as high as the signal‘s maximum frequency .

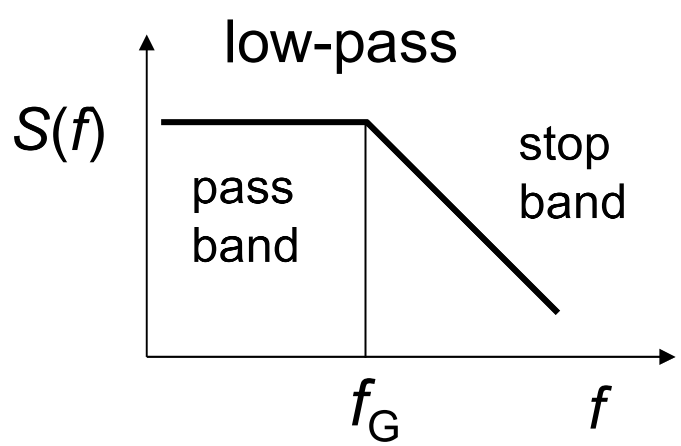
definitions **filtering**

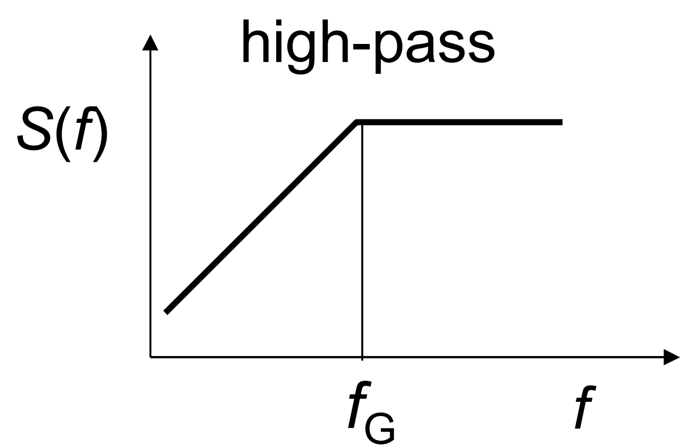
let f_G denote the **cut-off frequency**

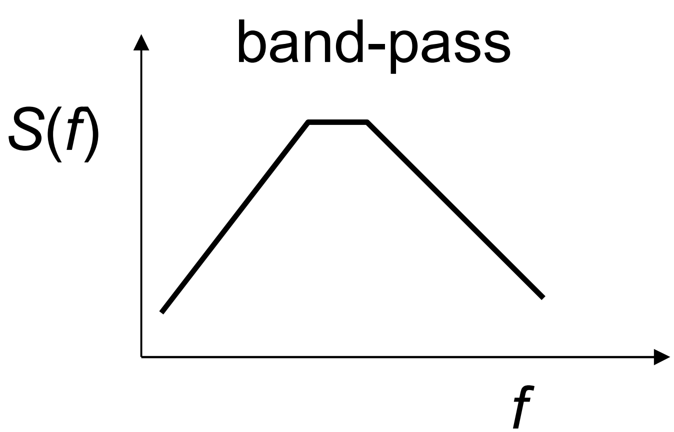
high-pass filter:
- all **frequency components smaller than f_G** are set to **0** (delete)
- all **frequency components larger than f_G** are mulitplied by **1** (allowed to pass through)

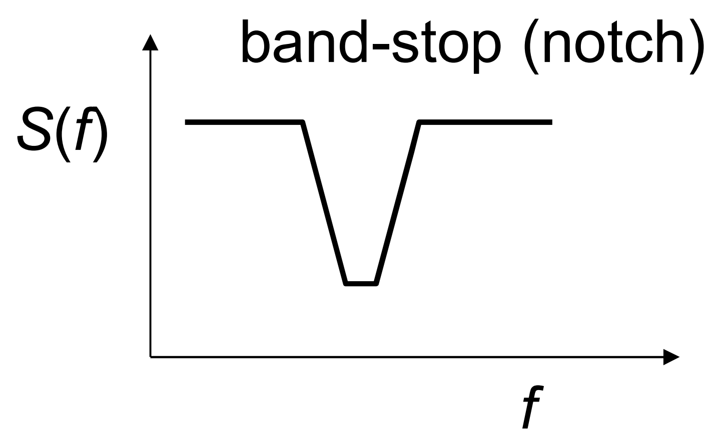
low-pass filter:
- all **frequency components larger than f_G** are set to **0** (delete)
- all **frequency components smaller than f_G** are multiplied by **1** (allowed to pass through)

definitions **filtering**



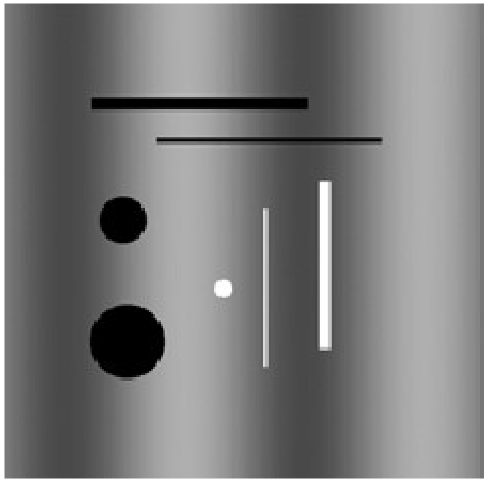






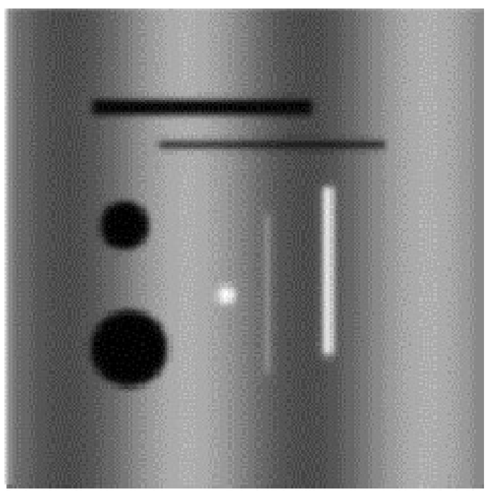
definitions **filtering**

original



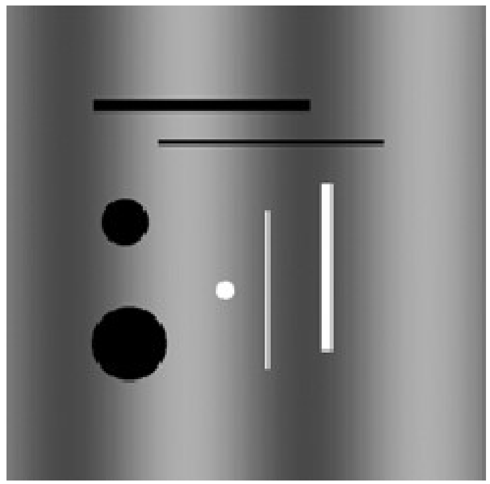
MTF(u,v) of
low-pass filter

low-pass
filtered

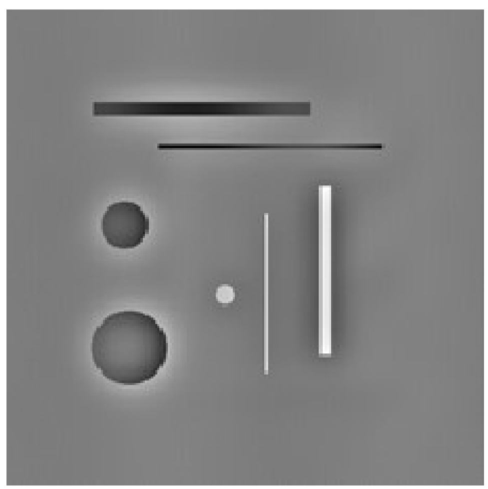


definitions **filtering**

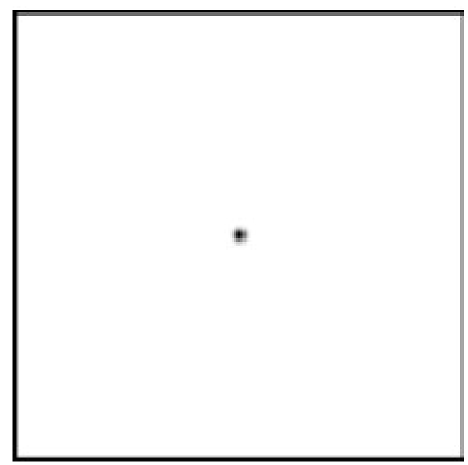
original



high-pass
filtered



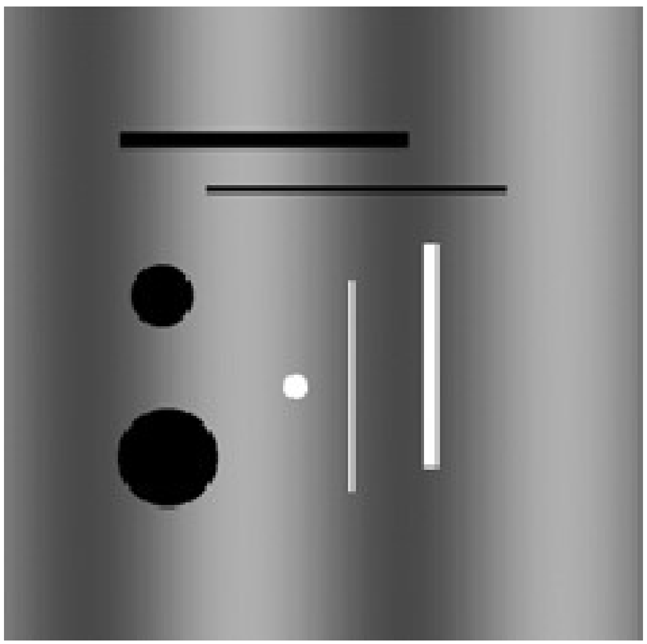
MTF(u,v) of
high-pass filter



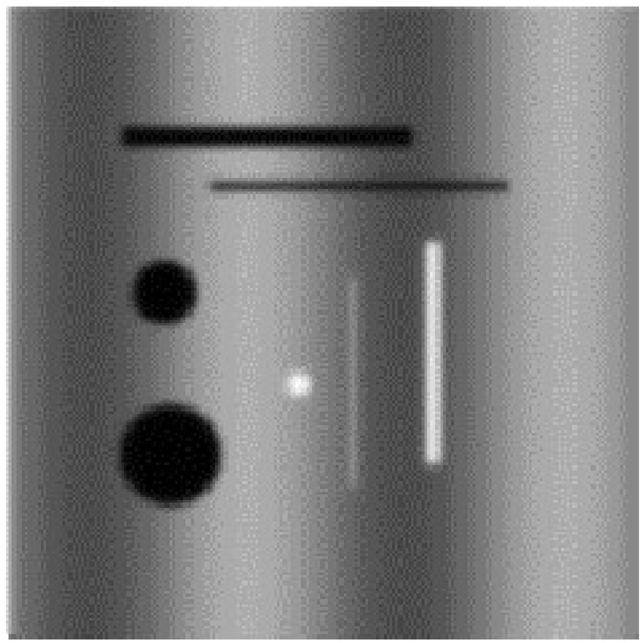
definitions

filtering

original

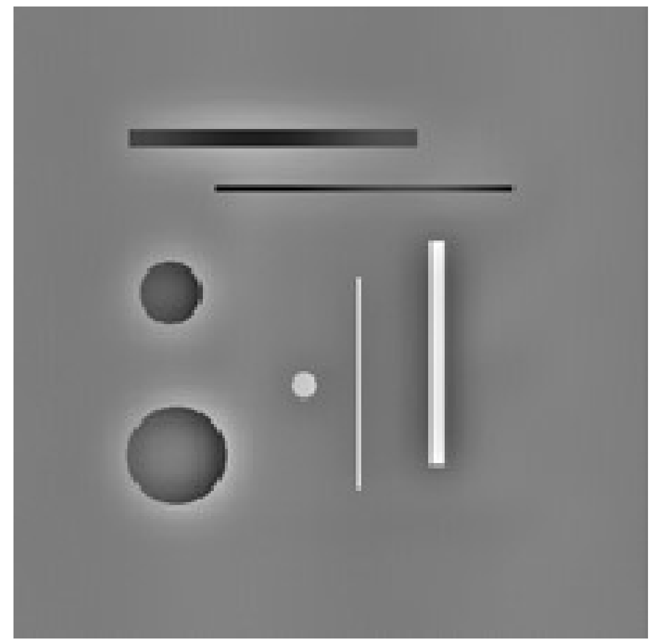


low-pass filtered



"smearing"

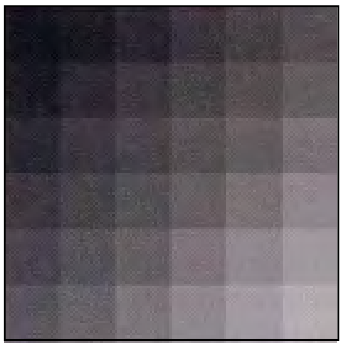
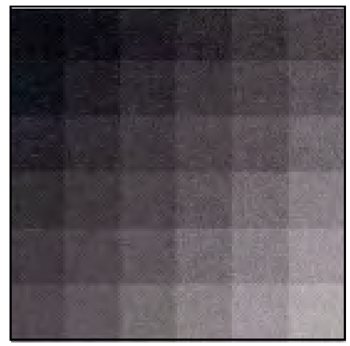
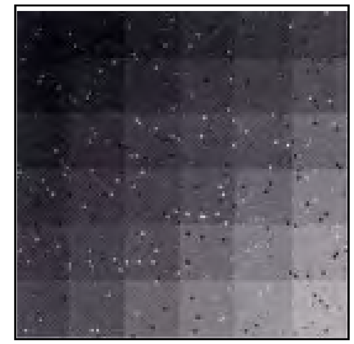
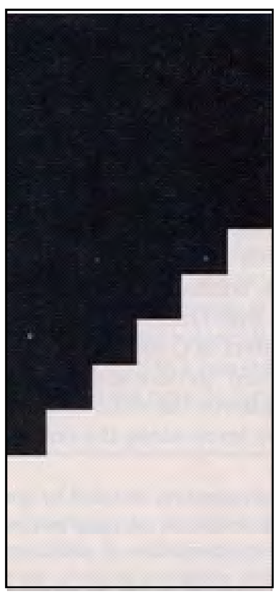
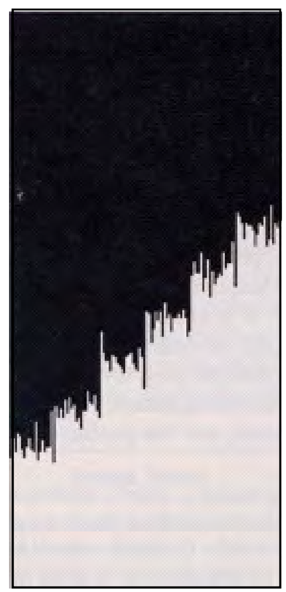
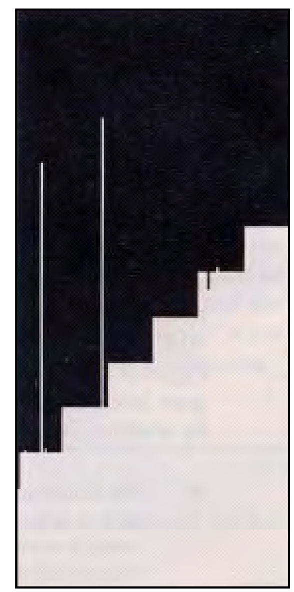
high-pass filtered



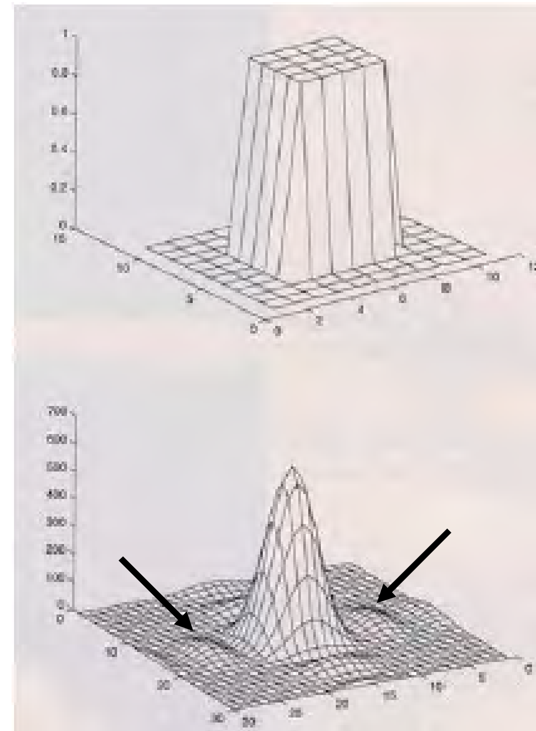
"edge enhancement"

definitions

filtering

	original	white noise (std. dev. = 5)	"salt and pepper" noise
synthetic checkerboard 120x120 grey levels			
grey level profile along one line			

definitions **filtering**



mean filter
smoothing through local averaging (low-pass)

assumption:
image has low-frequency content only
noise has high-frequency content only

spatial domain:
rectangular kernel
replace original pixel with weighted sum of neighboring pixel

caveat: produces echoes ("ringing") due to convolution with sin(x)/x

mean filter (3x3 low-pass)

original image
+ white noise

filtered
image

mean filter (3x3 low-pass)

original image
+ salt and pepper noise

filtered
image

definitions **filtering**

Gauss filter

extension of mean filter

replace rectangular with Gauss function

diminishes echoes

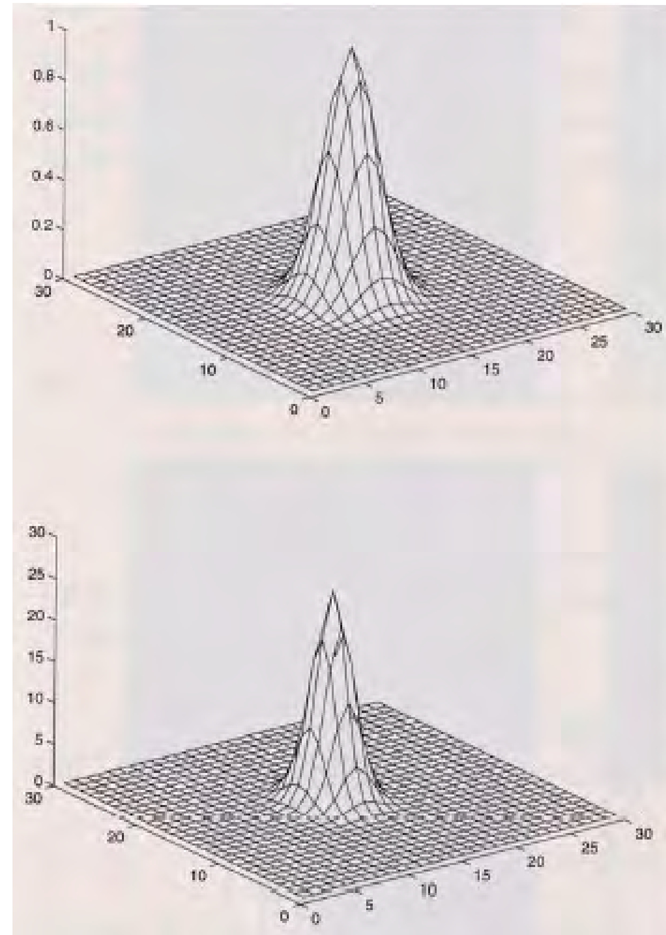
more advantageous than mean filter

(FT of Gauss function is Gauss function)

easy-to-implement; fast:

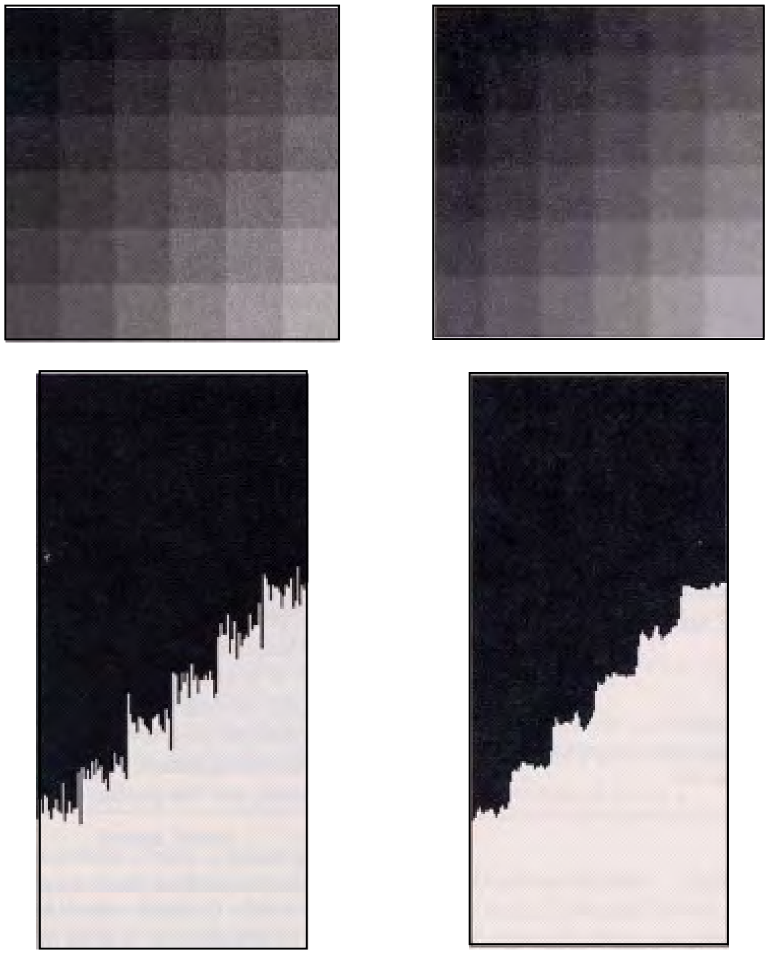
Gauss kernel can be decomposed:

2D-filter can be realized by two 1D-filter

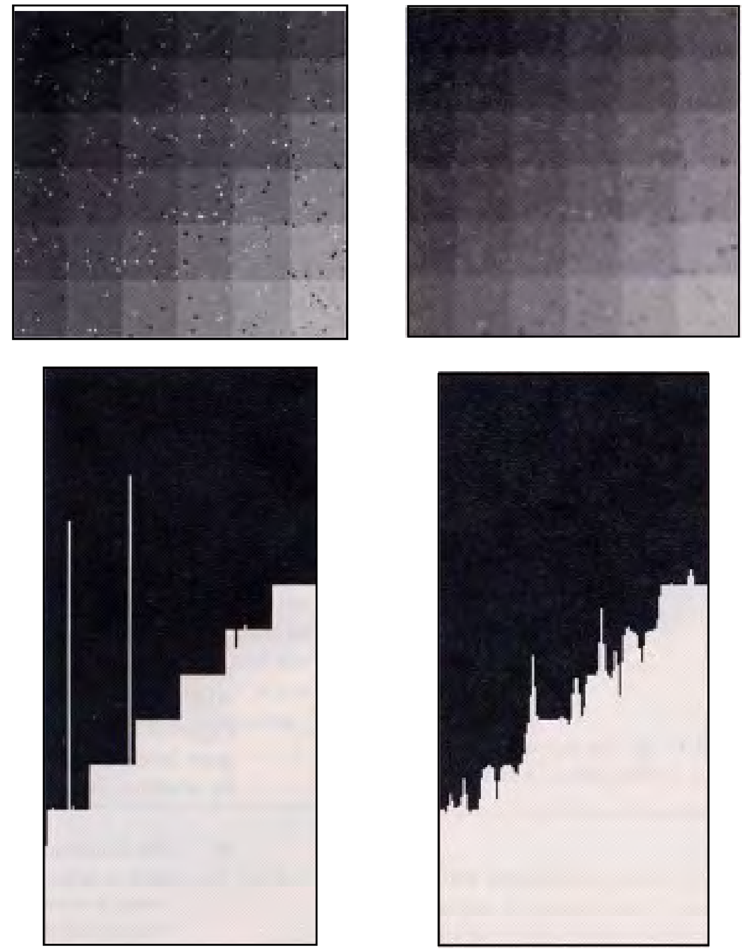


Gauss filter (kernel width: 5 pixel)

white noise
σ = 5)

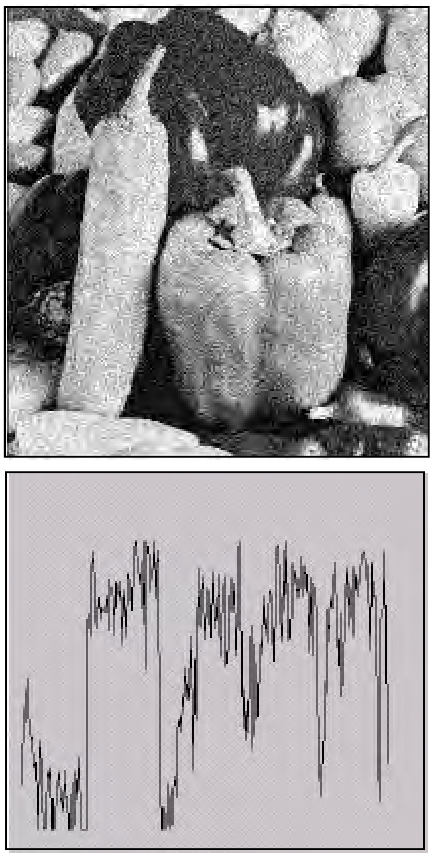


salt and pepper
noise

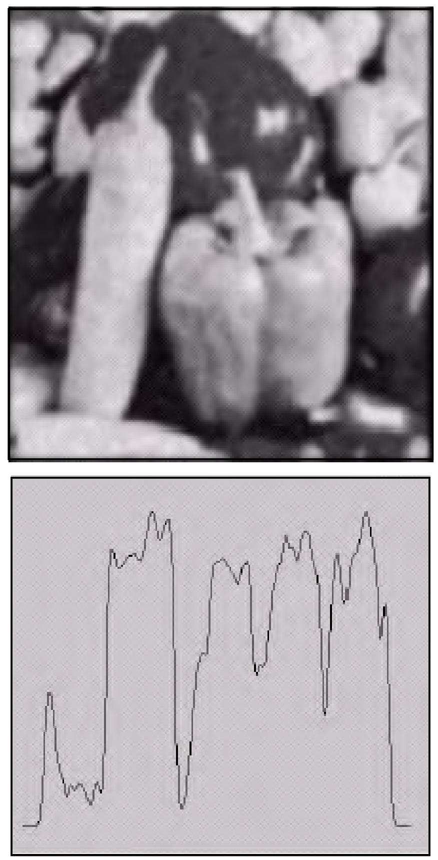


Gauss filter

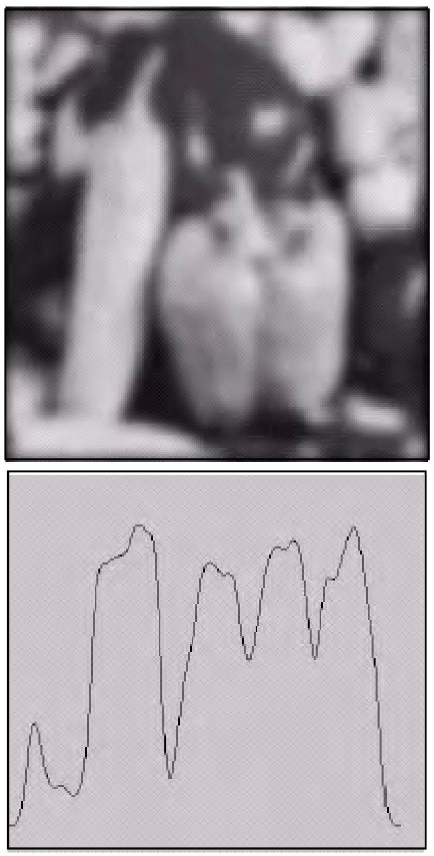
no filter



kernel width: 10 pixel

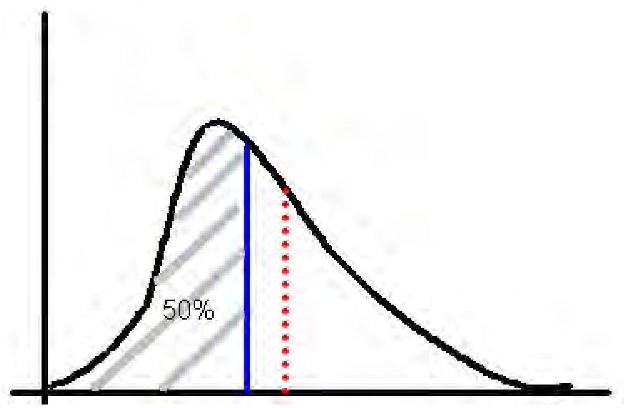


kernel width: 20 pixel



definitions

median



$\tilde{x}$ $\bar{x}$

median is the value separating the upper half from the lower half of a data sample

for discrete data:
- sort by size (rank order)
- then:

$$\tilde{x} = \frac{x_{\frac{N}{2}} + x_{\frac{N}{2}+1}}{2} \text{ for } N \text{ even}$$

$$\tilde{x} = x_{\frac{N+1}{2}} \text{ for } N \text{ odd}$$

- use median (or central value) if data sample not normally distributed
- median is insensitive to outlier

median filter (rank order filter)

for each Pixel $p(i,j)$ and its n x n neighborhood,

- sort pixel values by size
- replace $p(i,j)$ with median

22	17	102	105	106
21	21	100	99	102
19	22	20	102	102
24	21	97	101	104
19	18	101	108	101

⇒

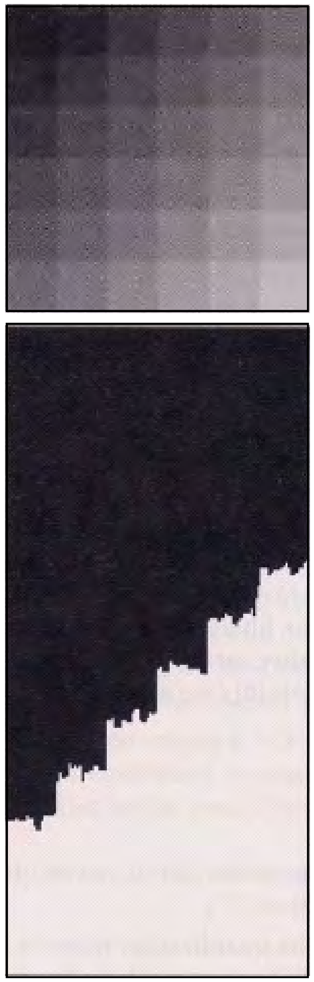
22	17	102	105	106
21	21	99	102	102
19	21	97	102	102
24	21	97	101	104
19	18	101	108	101

pros:
diminishes echoes
retains discontinuities
no influence of very noisy pixel

cons:
longer calculation times

median filter

white noise
(σ = 2)

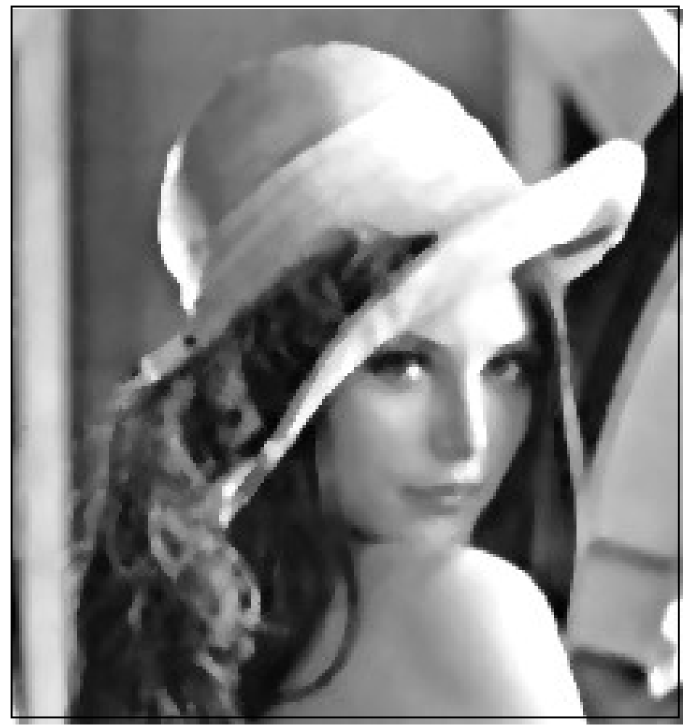


median filter
2 x 2
neighborhood

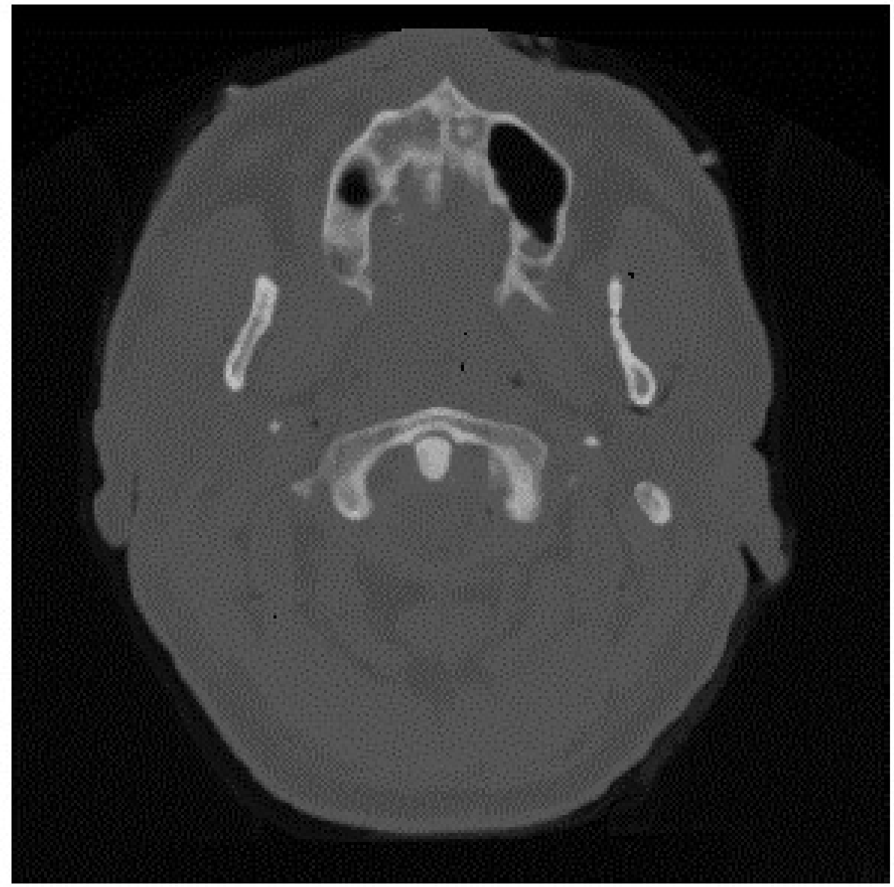
median filter

salt and pepper noise

median filter
with 2x2 neighborhood



median filter



summary (I)

prerequisite: imaging system is linear and translation invariant

(1) the mapping of some physical observable f(x,y) using an imaging system can be described mathematically as a convolution of the function f(x,y) with a function h(x,y) that fully characterizes the imaging system.
We have: $g(x,y) = f(x,y) * h(x,y)$, where g(x,y) denotes the output of the imaging system.

(2) the convolution theorem allows an equivalent description of the system in a reciprocal space (convolution in spatial domain corresponds to multiplication in Fourier space). We have: $G(u,v) = F(u,v) * H(u,v)$, where G, F, H denote the Fourier transforms of the functions f, g, h.

(3) in the spatial domain, h(x,y) is called **point spread function**, the function H(u,v) in the reciprocal domain is called **transfer function**

summary (II)

(4) important characterizing measures for an imaging system (in terms of …):

… ***spatial resolution***

- **point spread function** in spatial domain (sampling)
- **modulation transfer function** MTF(u,v) = |H(u,v)| in Fourier domain

… ***noise***

- **Detective Quantum Efficiency** DQE(u,v) = SNR_{input}/SNR_{output}
  (SNR = Signal-to-Noise Ratio)

(5) for an ideal system:

- point spread function = Dirac function (distortion-free system)
- MTF = 1 for all spatial frequencies
- DQE = 1 for all spatial frequencies

for a real system:

limited spatial resolution

- point spread function non-Dirac-like (broad)
- MTF decreases with higher spatial frequencies (artificial correlations in image)
- DQE < 1 (quantum noise, Poisson distribution)

summary (III)

(6) improving the signal-noise ratio (DQE) results in a diminished resolution (MTF) and vice versa.

(7) acquisition: obey the sampling (Nyquist) theorem to avoid aliasing errors:
sampling rate must be chosen at least twice the maximum spatial frequency in the object of interest

(8) tapering, if object not band-limited
(minimization of broad-band artificial contributions to image)

(9) post-processing of image:
- noise contaminations can be minimized by filtering the image
- choose filter appropriately! (the best filtering is no filtering)