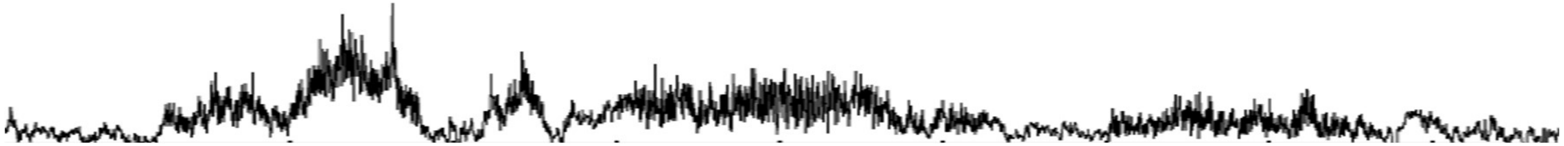
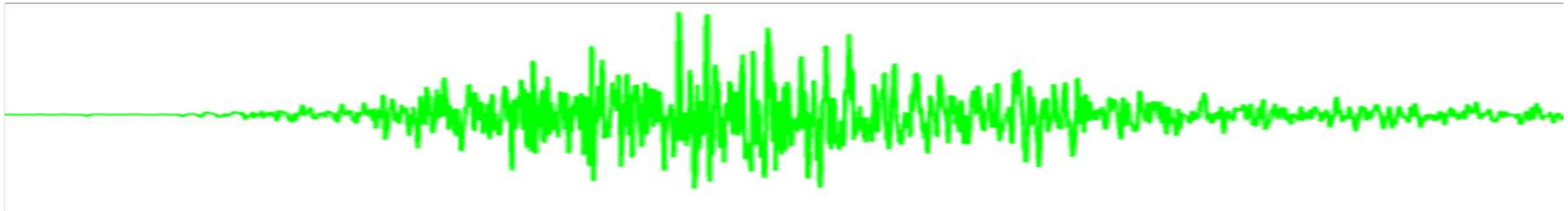
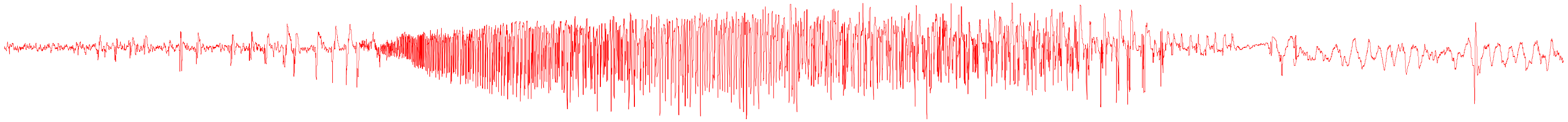
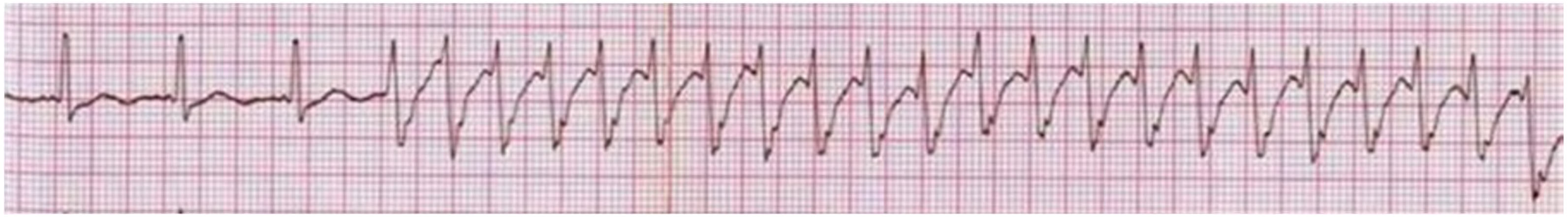
A brief Introduction to Dynamical Systems and Chaos Theory

Dynamical Systems









Motivation

mathematics:

dynamical system: is a system in which a *function* describes the *time dependence* of a point in some geometrical space

physics:

dynamical system: is described as a particle (or an ensemble) whose *state varies over time;* obeys differential equations involving time derivatives.

Making predictions about the system's future behavior requires an analytical solution of such equations or their integration over time through computer simulation.

Continuous-time dynamical systems

ordinary differential equation (first order)

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^d \; ; \;\; \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(t, \mathbf{x}(t), \beta)$$

terminology

$\mathbf{x}(t)$	state
$\mathbf{x}(0) = \mathbf{x}(t_0)$	initial state/condition
$x_1, x_2 \ldots, x_d$	state variables
d	dimension of system
β	control parameter
f	nonlinear function (in case of nonlinear system)
$\frac{\partial f}{\partial t} = 0$	autonomous system
$\frac{\partial f}{\partial t} \neq 0$	driven system

Continuous-time dynamical systems

ordinary differential equation (first order)

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^d \ ; \ \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(t, \mathbf{x}(t), \beta)$$

- given by some model
- find solution for specific initial condition (given control parameter)
  - often only possible numerically
  - often not of particular interest
- allows general statements about ensembles of solutions

- extensions: delay, stochastic, partial differential equations

Complexity of dynamical systems

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.

Robert M. May

Simple mathematical models with very complicated dynamics,
Nature 261 (1976)

some elementary examples in physics:

- classical pendulum / harmonic oscillator
- driven and damped pendulum
- celestial mechanics

Example: Lottka-Volterra Model (predator-prey dynamics)


A.J. Lottka


V. Volterra

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_1 x - a_2 x y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -a_3 y + a_4 x y$$

x: number of prey (e.g. hare)
y: number or predator (e.g. lynx)
$a_1,\ldots a_4$: positive real parameter describing interaction between species

observation

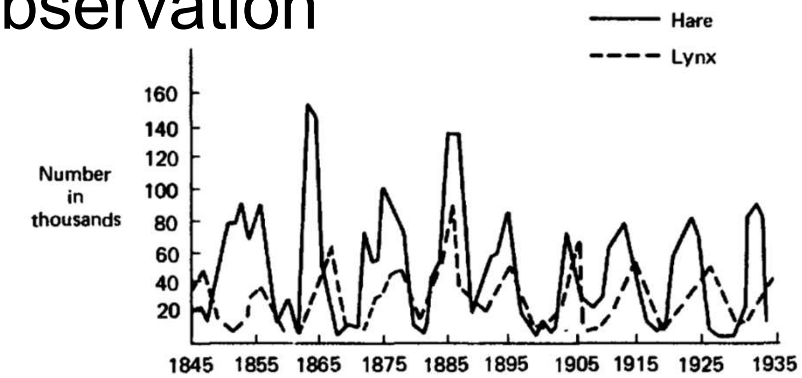
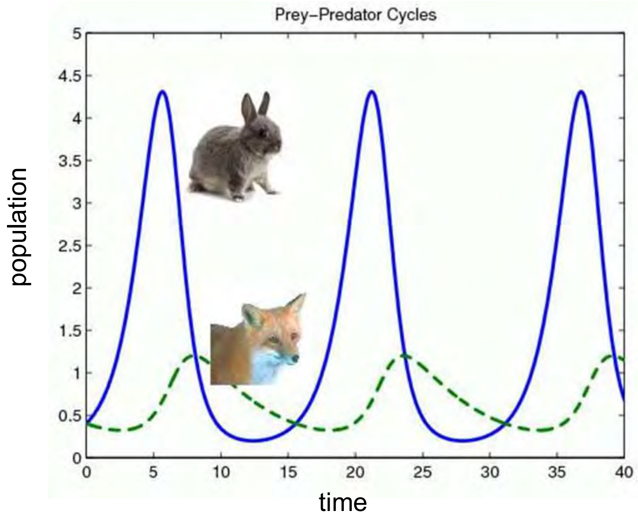


Figure 48-1 Oscillation observed in Canada of populations of lynx and hare (*data from* E. P. Odum, *Fundamentals of Ecology*, Philadelphia: W. B. Saunders, 1953).

model data



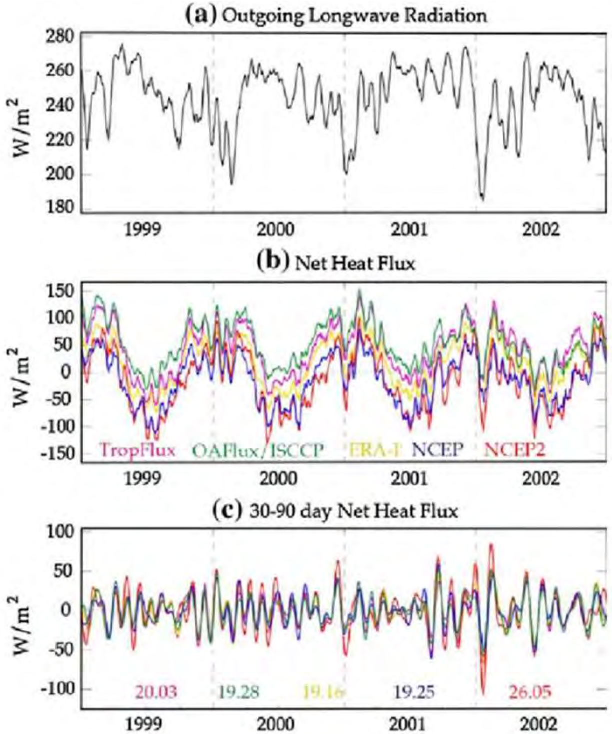
from: www.scholarpedia.org/article/Predator-prey_model

Example: Lorenz oscillator

E.N. Lorenz

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y-x) \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= x(\rho-z)-y \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= xy-\phi z
\end{aligned}
$$

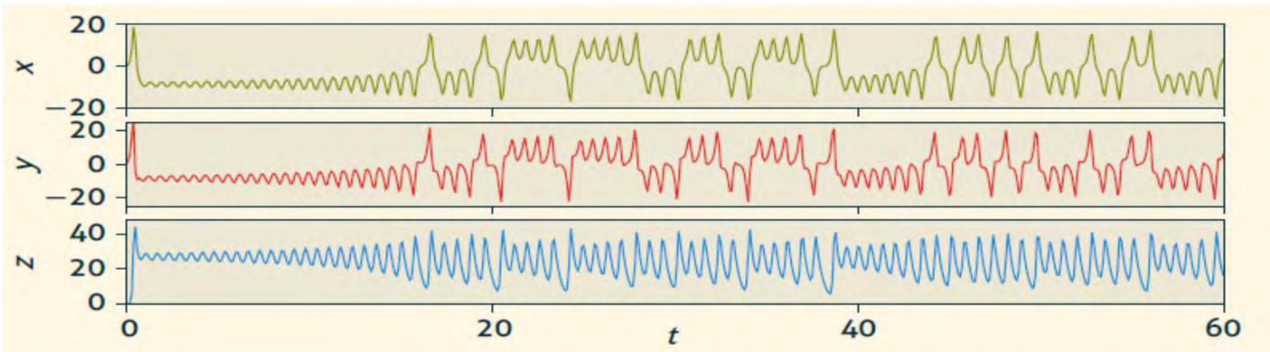
observation



from: Kumar et al., Clim. Dyn. 38, 1521, 2012

- simple model for atmospheric convection
- butterfly effect:
  does the flap of a butterfly's wings in Brazil set off a tornado in Texas?

model data



from: G. Ansmann

Chaotic dynamics (tentative definition)

- sensitive to initial conditions (butterfly effect)

- qualitatively recurring

but not:

- periodic

- stagnant

- “escalating”

Discrete-time dynamical system

iterative map

$$\mathbf{x}_t \in \mathbb{R}^d \ ; \ \ \mathbf{x}_{t+1} = F(t, \mathbf{x}_t, \beta)$$

- easier to analyze
- easier to simulate
- every ODE can be transformed to a map
  (e.g., via numerical integration or Poincaré sections)

Example: logistic map

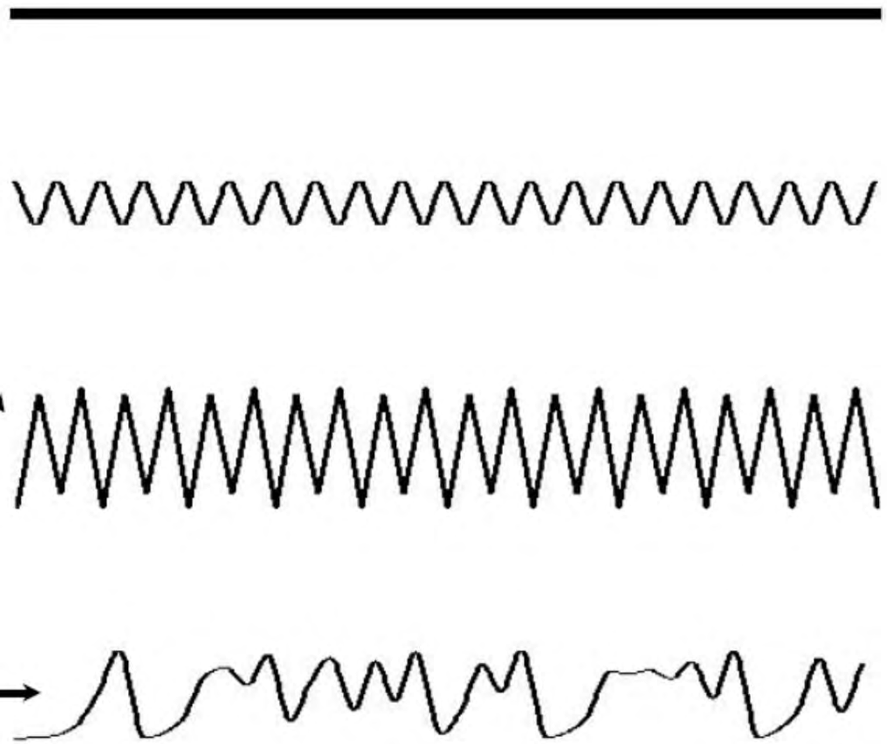

P.F. Verhulst


R. May

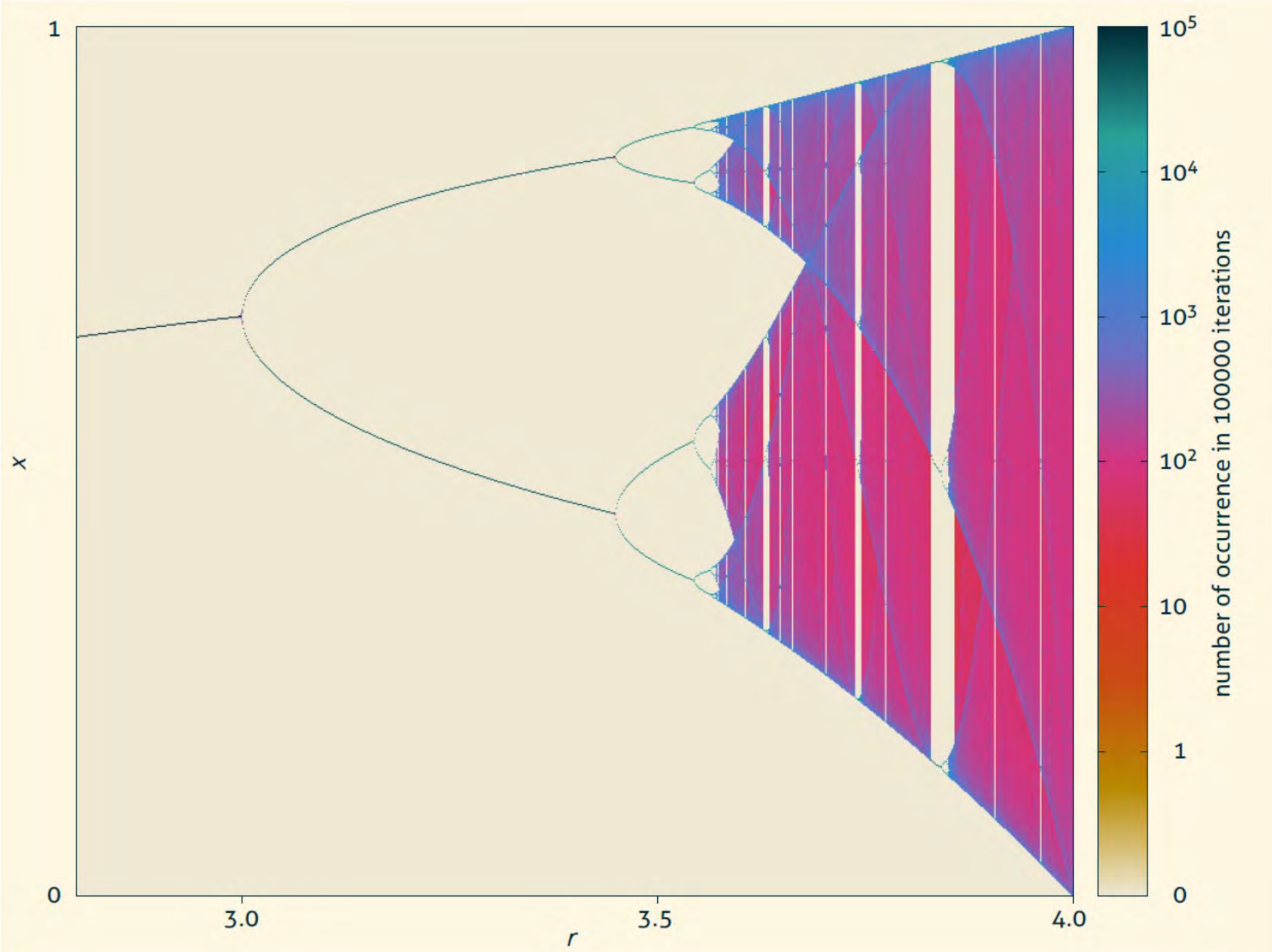
model for population growth (1837)

$$x_{n+1} = r x_n (1 - x_n); \quad x_n \in [0, 1]; \quad r \in [0, 4]$$

P	Periode (2^P)	r
0	1	< 3,0
1	2	3,0
2	4	3,449490
3	8	3,544090
4	16	3,564407
5	32	3,568750
6	64	3,569690
7	128	3,569890
8	256	3,569934
9	512	3,569943
10	1024	3,5699451
11	2048	3,569945557
?	Chaos	3,569945672

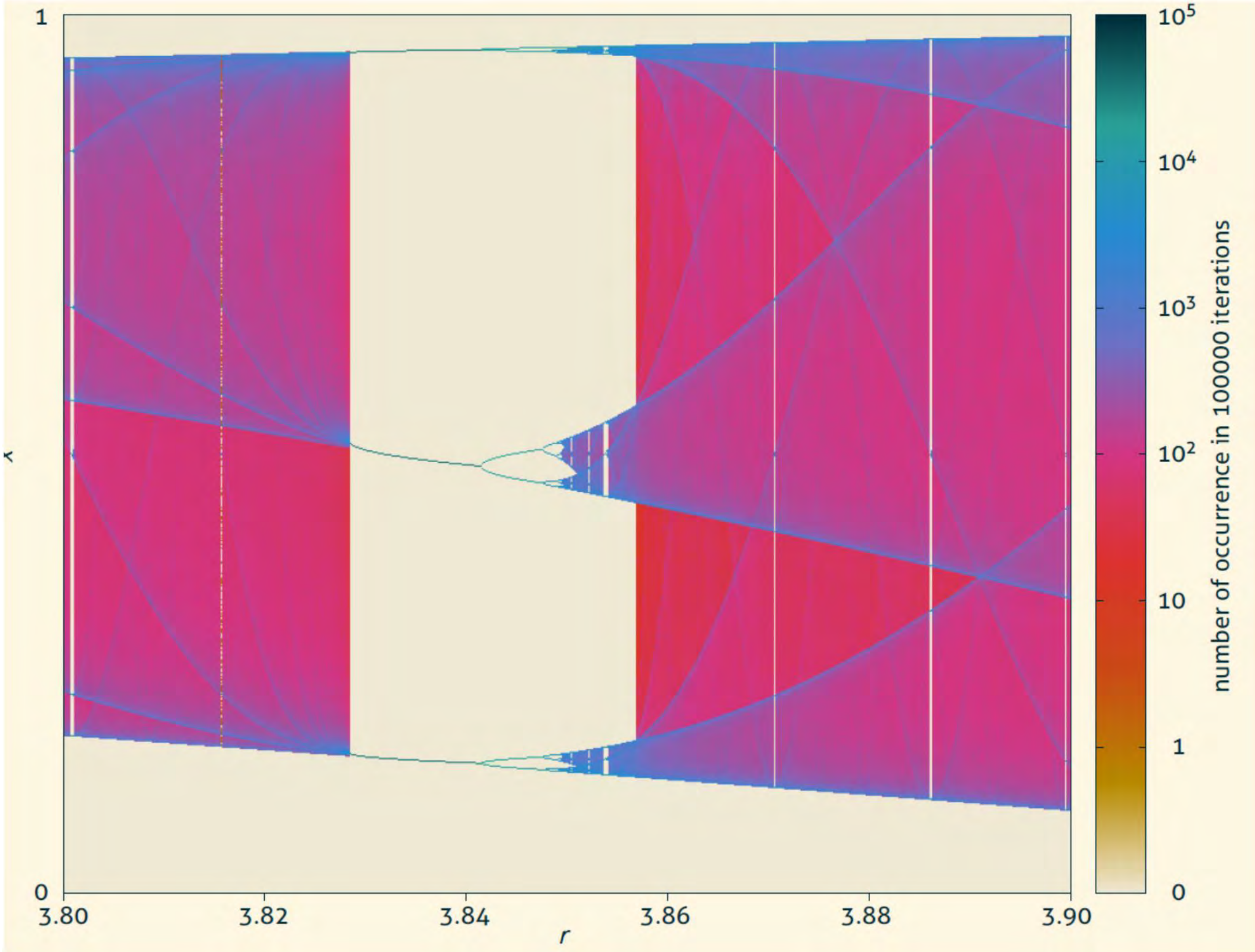


Example: logistic map



from: G. Ansmann

Example: logistic map



from: G. Ansmann

Example: logistic map

The Feigenbaum constant:

let r_1, r_2, … denote the values of control parameter r at which bifurcations happen.
We find:

$$\lim_{i \to \infty} \frac{r_i - r_{i-1}}{r_{i+1} - r_i} = \delta \approx 4.669$$

- δ: Feigenbaum constant
- universal for many similar processes
- also found in natural systems

  turbulent cascade in fluids
  nonlinear oscillations in electric circuits
  nonlinear oscillations in chemical reactions (Belousov-Zhabotinskii reaction)
  heart: ventricular fibrillation (lethal)

Phase space

ordinary differential equation (first order)

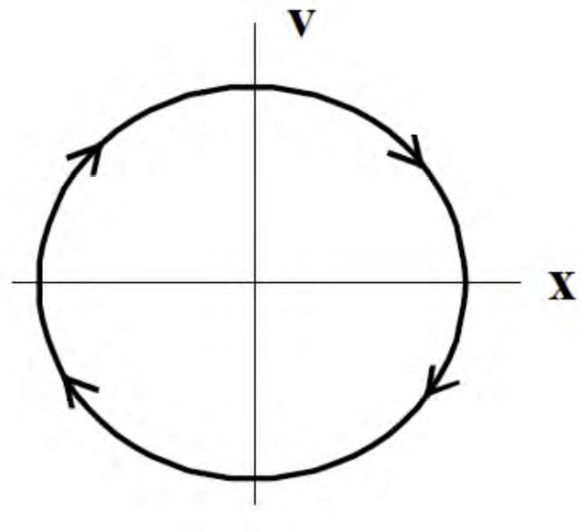
$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^d \ ; \ \ \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(t, \mathbf{x}(t), \beta)$$

Phase space:

representation of states/trajectories ($\mathbf{x}$) of the dynamics in d-dimensional space.

- time is only implicit
- also known as *state space*

Example:
harmonic oscillator



Invariant sets

Time-evolution function:

$$\phi_{f,\tau} : \mathbb{R}^d \to \mathbb{R}^d \ ; \phi_{f,\tau}(\mathbf{x}(t)) = \mathbf{x}(t+\tau)$$

for any solution $\mathbf{x}$ of $\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(\mathbf{x}(t), \beta)$ analogously for maps

Forward-invariant set / manifold:

$\mathcal{S}_f \subset \mathbb{R}^d$ is a non-empty set for which holds:

$$\forall \tau > 0 : \mathcal{S}_f = \phi_{f,\tau}(\mathcal{S}_f) := \{\phi_{f,\tau}(x) | x \in \mathcal{S}\}$$

Irreducible invariant set:

An invariant set without an invariant true subset

Attractors

$\mathcal{A}_f \subset \mathbb{R}^d$ denotes a set for which holds:

- $\mathcal{A}_f$ is an irreducible forward-invariant set
- there exists a neighborhood $\mathcal{B}_{\mathcal{A}_f} \supset \mathcal{A}_f$ such that

$$\lim_{\tau \to \infty} \phi_{f,\tau}(\mathcal{B}_{\mathcal{A}_f}) = \mathcal{A}_f$$

- the maximal $\mathcal{B}_{\mathcal{A}_f}$ is called *basin of attraction*
- dynamics within $\mathcal{B}_{\mathcal{A}_f} \cap \mathcal{A}_f$, i.e., motion onto the attractor, is called *transient*
- more elaborate definitions to handle pathological cases

Attractors

- if the system dynamics is confined to a certain region in phase space, then this region is called *attractor*

- set of all solutions of the system's dynamical equations

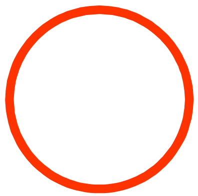
- three kinds of (irreducible) invariant sets / attractors important to this course:

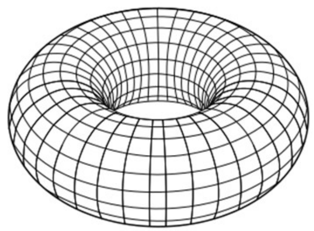
• fixed points

• periodic

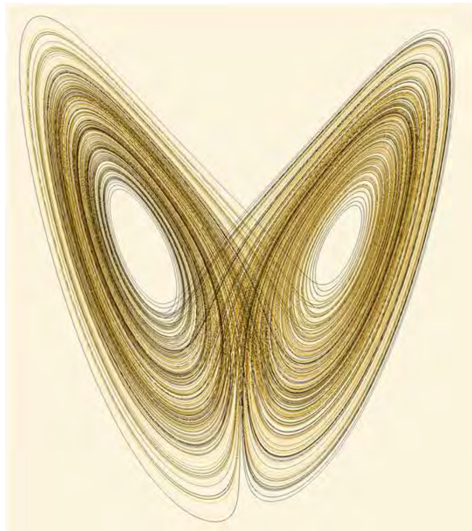
simple periodic / limit cycle

quasiperiodic / torus, hypertorus





• strange / chaotic / fractal



Lorenz attractor

each type corresponds to a different type of dynamics

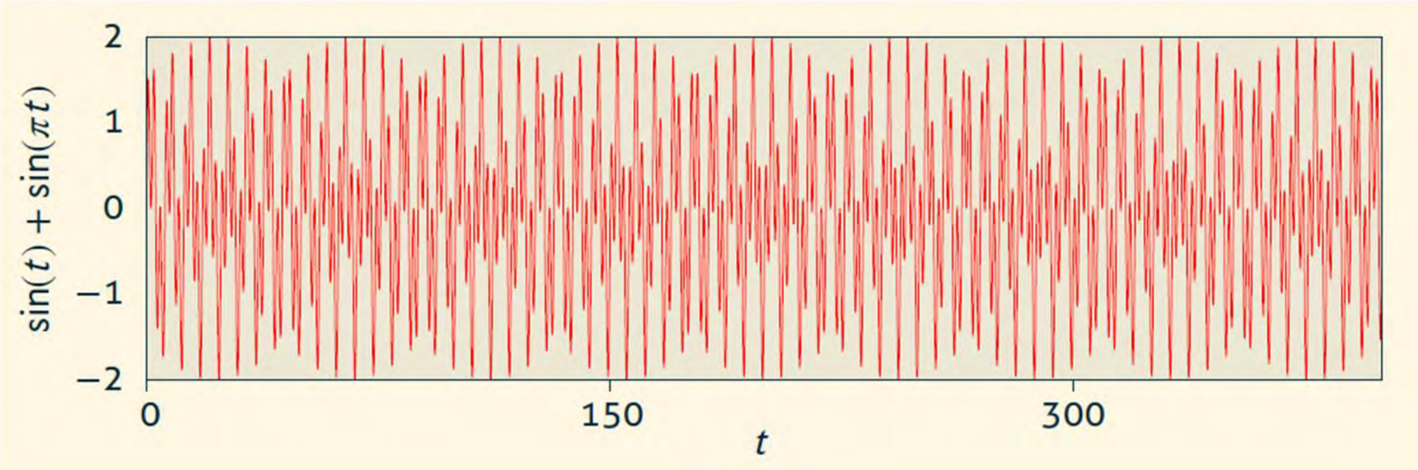
Intermezzo: Quasiperiodicity

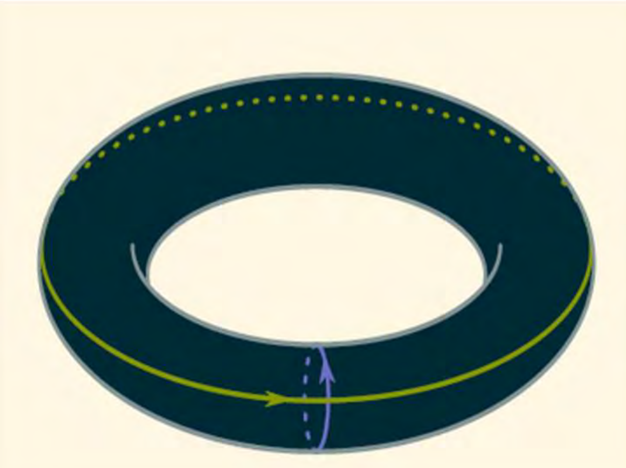
Incommensurability

two numbers a and b are incommensurable iff $\frac{a}{b} \notin \mathbb{Q}$

Quasiperiodicity

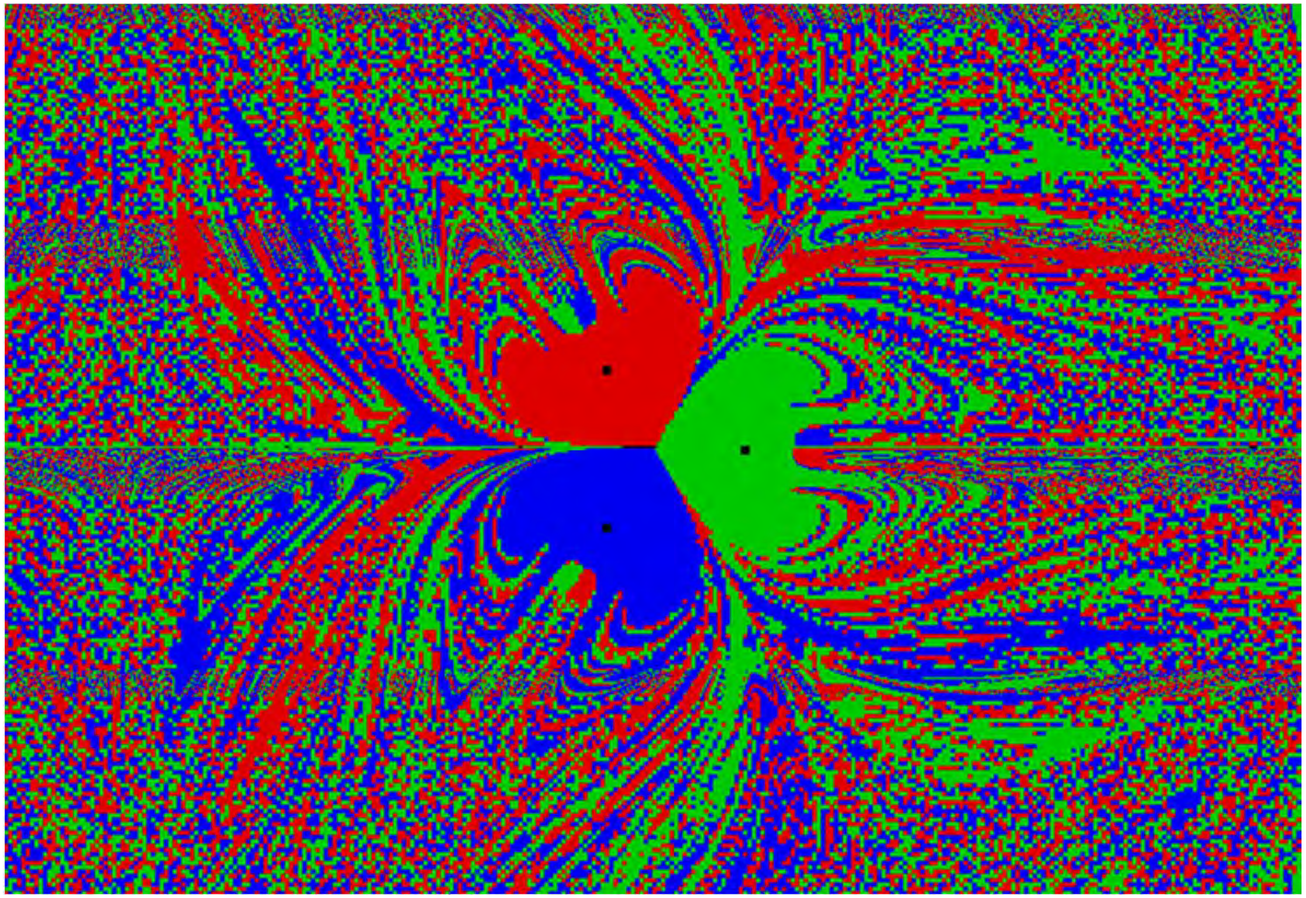
superposition / combination of two (or more) periodic processes with incommensurable frequencies





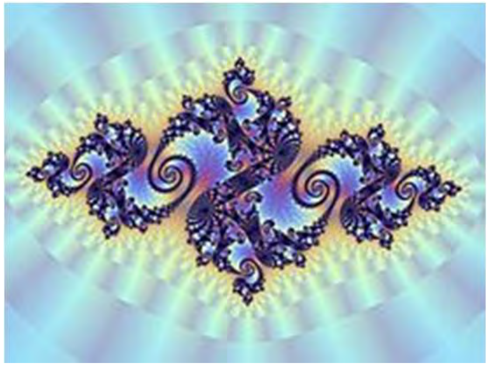
from: G. Ansmann

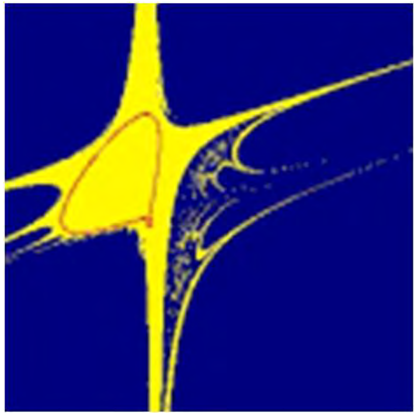
Intermezzo: Attractor Basins magnetic pendulum

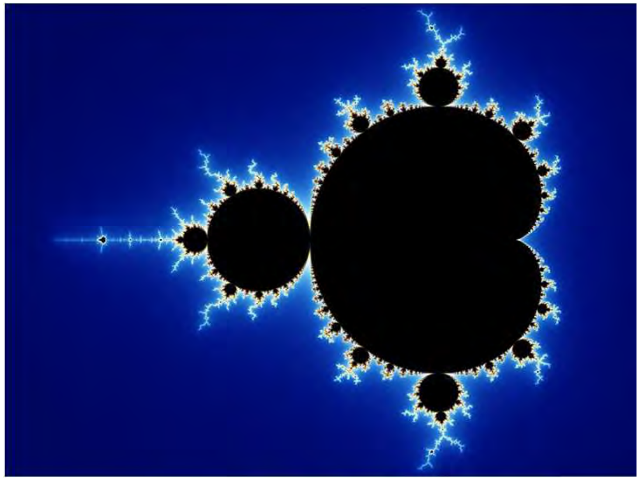


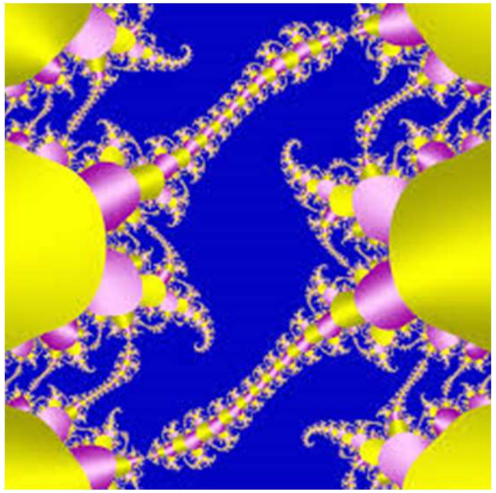
Intermezzo: Attractor Basins

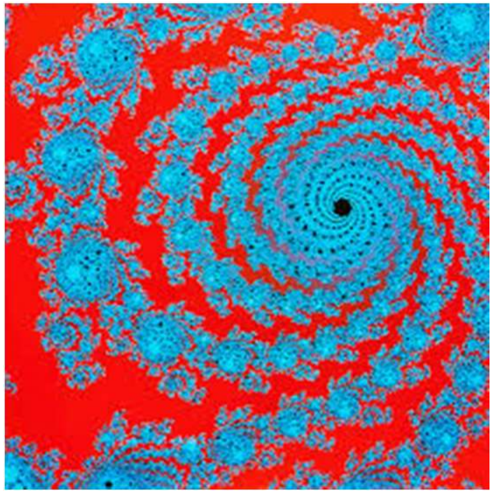
Julia sets

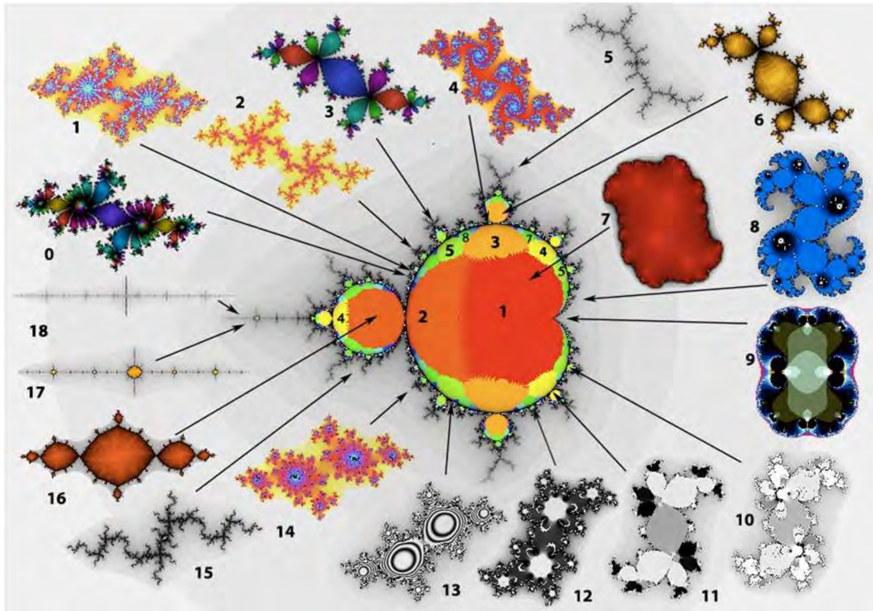












Characterizing fixed points

for continuous, differentiable maps ($\mathbf{x}_{t+1} = F(t, \mathbf{x}_t, \beta)$), we have:

z is fixed point of F

$\Leftrightarrow$ $\{z\}$ is invariant set

$\Leftrightarrow$ $F(z) = z$

from local linearization and continuity of F, we have:

z is stable fixed point of F

$\Leftrightarrow$ $\{z\}$ is attractor

$\Leftrightarrow$ $F(z) = z$ and $|F'(z)| < 1$

Characterizing fixed points

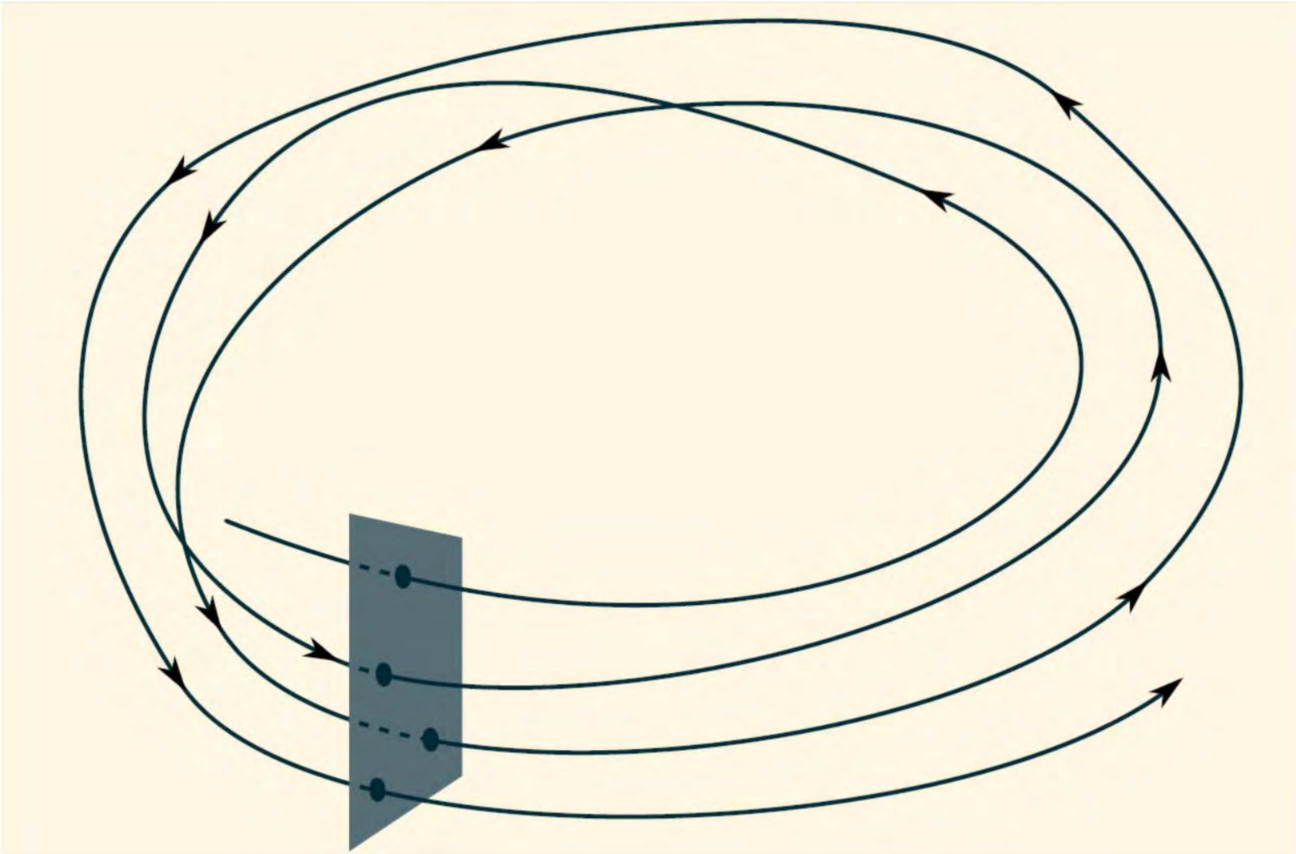
for ordinary differential equations ($\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(t, \mathbf{x}(t), \beta)$), we have:

z is fixed point of f
$\Leftrightarrow$ $\{z\}$ is invariant set
$\Leftrightarrow$ $f(z) = 0$

from local linearization and continuity of f, we have:

z is stable fixed point of f
$\Leftrightarrow$ $\{z\}$ is attractor
$\Leftrightarrow$ $f(z) = 0$ and ∇f has no eigenvalue μ with $\Re(\mu) > 0$

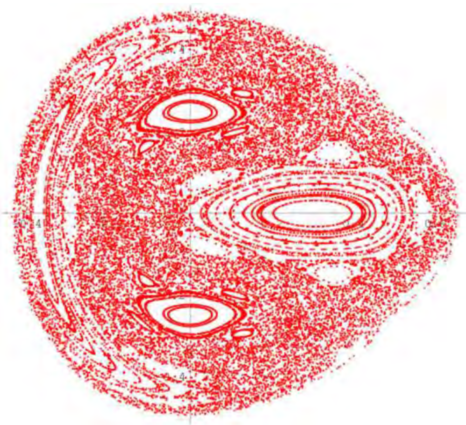
Poincaré sections



limit cycle:
→ point

torus:
→ limit cycle

chaotic motion:
→ complex structures



Poincaré sections

Let $s_0 < s_1 < \ldots$ denote the times of "marker events", e.g.:

- intersections of the trajectory with a plane (in a given direction)
- local extrema of some observable
- given phase of driving oscillation

Poincaré map

the map that yields the sequence $\mathbf{x}(s_0), \mathbf{x}(s_1), \ldots$

- may distill relevant aspects of the dynamics
- simplifies dynamics
- allows to estimate stability of periodic solutions
- may be difficult to determine

Classifying dynamical systems via divergence

consider f as a vector field in phase space

→ f describes the phase-space flow

→ $\nabla\cdot f$ describes expansion/contraction of infinitesimal phase-space volume $\mathrm{V(t)}$ under f, i.e., time evolution (Liouville's theorem)*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}|V(t)| = |V(t)|\,\nabla\cdot f(V(t))$$

Classifying dynamical systems via divergence

Three kinds of dynamics:

divergence	name	attractors
$\nabla \cdot f = 0$	conservative	no (Liouville)
$\nabla \cdot f > 0$	unstable	no
$\nabla \cdot f < 0$	dissipative	yes

(assuming constant sign along the trajectory)

Divergence and Lyapunov exponents

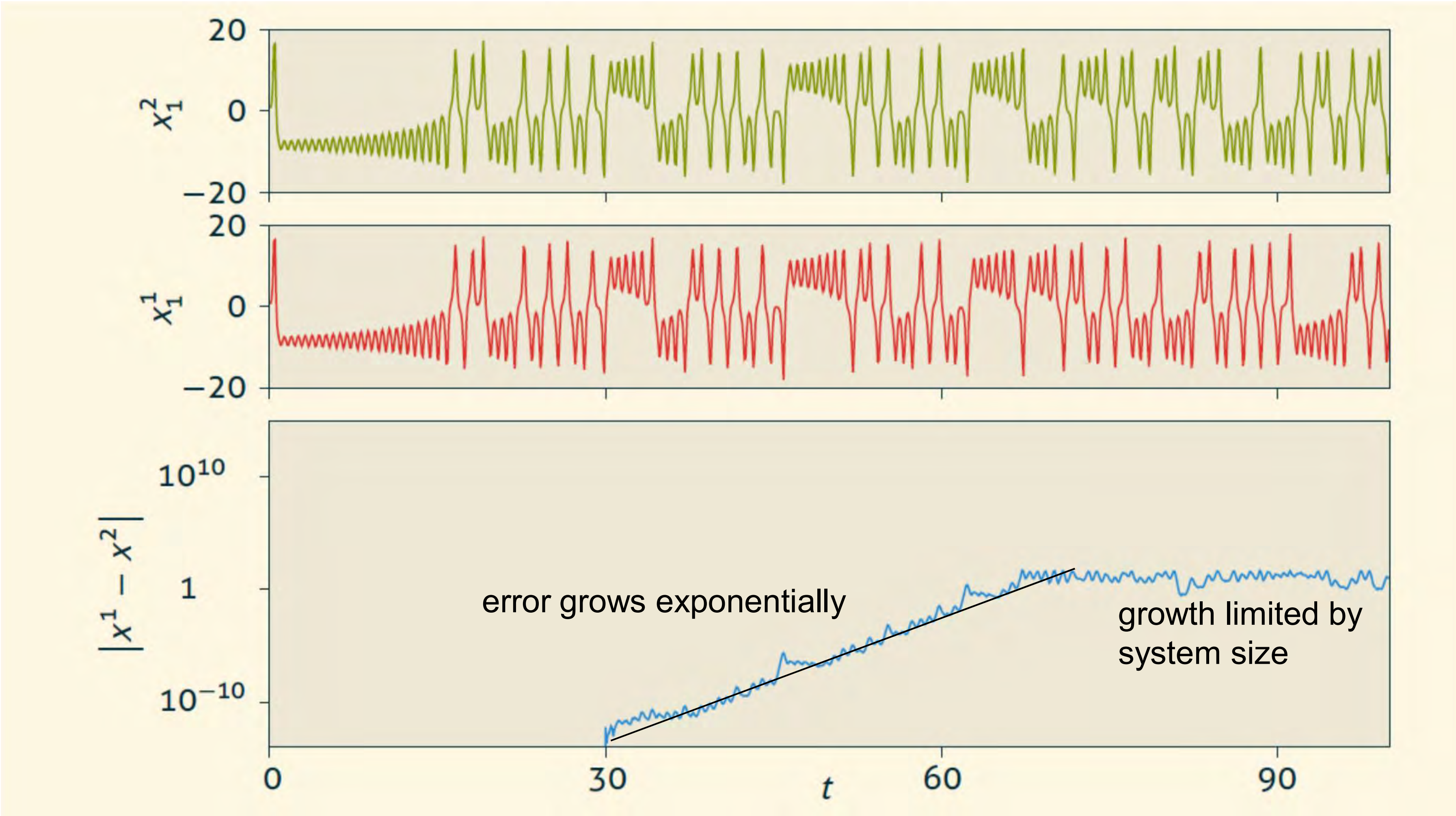
- divergence quantifies growth of volumes

$$V(t) = V(t_0)\,\mathrm{e}^{(\nabla \cdot f)t}$$

- Lyapunov exponents quantify growth of "vectors"

Lyapunov exponents

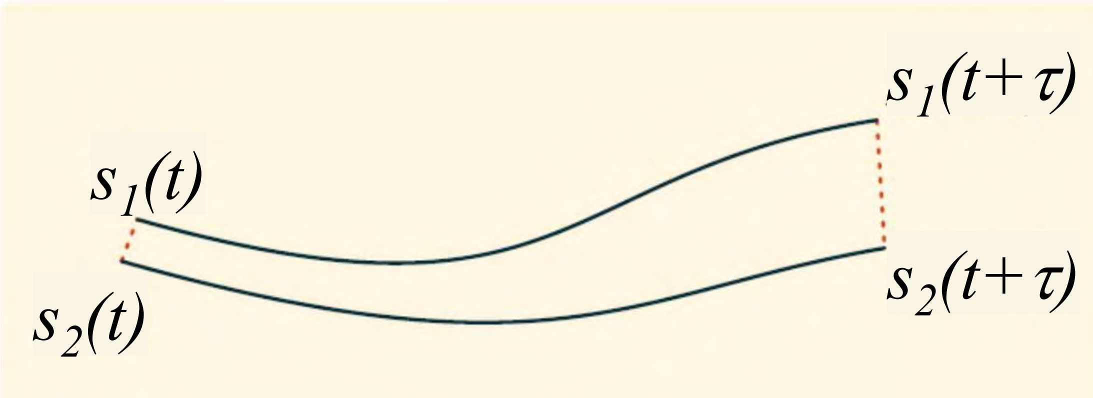
Example: two identical Lorenz oscillators with initial conditions; one oscillator is slightly perturbed (10^{-14}) at t = 30



from: G. Ansmann

Largest Lyapunov exponent

Consider evolution of two nearby trajectory segments s_1 and s_2



For infinitesimally close trajectory segments $(|s_1(t) - s_2(t)| \to 0)$
and for infinite time evolution ($\tau \to \infty$)
the distance between segments grows or shrinks exponentially:

$$|s_1(t+\tau) - s_2(t+\tau)| = |s_1(t) - s_2(t)|\, \mathrm{e}^{\lambda_1 \tau}$$

Largest Lyapunov exponent — Definition

$$|s_1(t+\tau) - s_2(t+\tau)| = |s_1(t) - s_2(t)|\, \mathrm{e}^{\lambda_1 \tau}$$

Solve for λ_1 and implement the limits.

Let s_1 and s_2 denote two near trajectory segments of the dynamics. The first Lyapunov exponent is defined as:

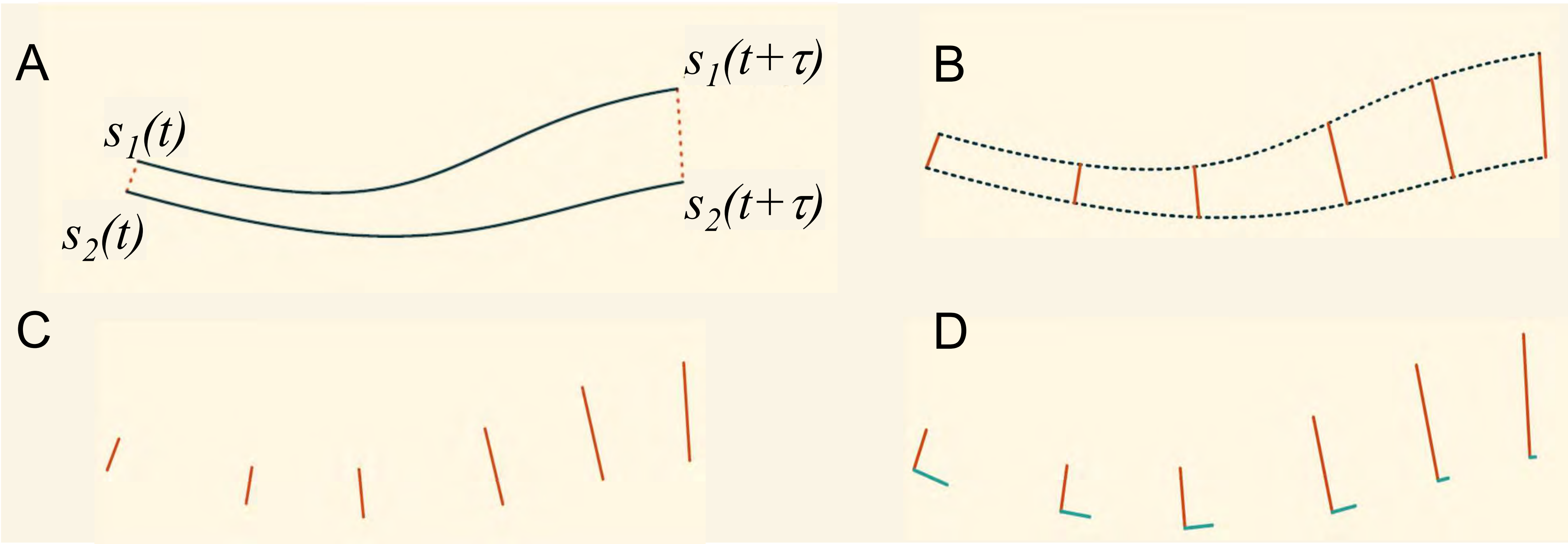
$$\lambda_1 := \lim_{\tau \to \infty} \lim_{|s_1(t)-s_2(t)| \to 0} \frac{1}{\tau} \ln\left(\frac{|s_1(t+\tau)-s_2(t+\tau)|}{|s_1(t)-s_2(t)|} \right)$$

Also: *largest Lyapunov exponent* or just *Lyapunov exponent*.

Further Lyapunov exponents

A perturbation aligns itself along the ***direction*** of strongest expansion / weakest contraction.

- this takes some time (A and B)
- direction depends on the current state (C)
- orthogonal directions for further Lyapunov exponents (D)



Lyapunov spectrum

Second Lyapunov exponent

The largest Lyapunov exponent determined with perturbations orthogonal to the direction corresponding to the first Lyapunov exponent

Third Lyapunov exponent

The largest Lyapunov exponent determined with perturbations orthogonal to the directions corresponding to the first two Lyapunov exponents

…

- practical: frequent orthogonalizations
- as many Lyapunov exponents as phase-space dimensions:

$$\sum_{i=1}^{d} \lambda_i = \nabla \cdot f$$

Lyapunov spectrum and type of Dynamics

For bounded, continuous-time dynamical systems, we have:

signs of Lyapunov exponents	dynamics
−, −−, −−−, …	fixed point
+, ++, +++, …, +0, ++0, …	not possible (unbounded)
0, 00, 000, …	no dynamics ($f = 0$)
0−, 0−−, 0−−−, …	periodic / limit cycle
00−, 00−−, 00−−−, …	quasiperiodic (torus)
000−, 0000−, …, 000−−, …	quasiperiodic (hypertorus)
+0−, +0−−, +0−−−, …	chaos
++0−, +++0−, …, ++0−−, …	hyperchaos
∞, …	noise

Deterministic Chaos

No commonly accepted definition.

For our purposes:

"*A bounded, deterministic dynamics with a positive Lyapunov exponent*."

The exponential divergence or convergence of nearby trajectories (Lyapunov exponents) is conceptually the most basic indicator of deterministic chaos.

M. Sano and Y. Sawada
Measurement of the Lyapunov spectrum from a chaotic time series, PRL 55 (1985)

Properties of Deterministic Chaos

Necessary conditions:

- in continuous-time dynamical systems: three dimensions or more
- non-linearity

Properties:

- sensitivity to initial conditions (butterfly effect)
  → only predictable on a short time scale
- no regularity
- fractal or strange attractors
- many more (→ future lectures)

So far:

- dynamical equations of motion known and fixed
- almost arbitrarily long time series (through simulation)
- high (unlimited) precision
- access to all dynamical variables
- no noise

Typical experimental situation:

- dynamical equations of motion unknown with changing parameters
- short time series
- low (limited) precision
- access to few (or only one) dynamical variables
- noise and uncertainties